A New Hybrid Dai-Yuan and Hestenes-Stiefel Conjugate Gradient Parameter for Solving System of Nonlinear Equations

[1] Jamilu Sabi'u, Northwest University Kano, Nigeria, sabiujamilu@gmail.com

[2] Mohammed Yusuf Waziri, Bayero University Kano, Nigeria, mywaziri@gmail.com

[3] Abba Idris, Northwest University Kano, Nigeria, abbaidris014@gmail.com

Abstract **:- In this research work, we suggest a new hybrid conjugate gradient parameter for solving the system of nonlinear equation by the combination of the Dai-Yuan and Hestenes-Stiefel conjugate gradient parameters. The new hybrid conjugate gradient parameter of the proposed method gives it the advantage to solve relatively large-scale problems (300,000- variables) with lower storage requirement compared to some existing methods (conjugate gradient parameter). Under the appropriate conditions, numerical results on benchmark test problems show that the proposed method is practically effective (less number of iterations and computer time).**

Keywords: **Line search, Systems of nonlinear equations, Conjugate gradient method.**

1. INTRODUCTION

We consider the nonlinear systems of equations

$$F(x) = 0, \tag{1}$$

where $F : R^n \to R^n$ is a continuously differentiable mapping, the Line search methods are one of the well-known techniques for solving equation (1); see [2, 11]. At the k^{th} iteration, the new point is introduced by

$$x_{k+1} = x_k + \alpha_k d_k, \tag{2}$$

where the search d_k direction and $\alpha_k > 0$ is computed by exact or inexact line search rules. More recently, waziri and sabi'u [11] proposed a derivative-free conjugate gradient method and its global convergence for solving symmetric nonlinear equations based on non-monotone line search methods; numerical results showed their method is promising. Notwithstanding, Cheng and Li [4] extended the non-monotone line search method proposed by Zhang and Hager [13] to the spectral residual method to solve large-scale nonlinear systems of equations. Moreover, Zhang and Zhou [14] developed a spectral gradient projection method by combining the spectral gradient method in [3] with the projection method in [8]. Wang et al. [10] presented a projection method for nonlinear equations with convex constraints. Also recently, Goglin and Maojun [6] derived a three-terms Polak-Ribi e' re–Polyak conjugate gradient algorithm for large-scale nonlinear equations using projection approach. Numerical experiments showed that each over mentioned method performs quite efficient, some of recent methods for solving system of nonlinear equations included [18,19,20]. Motivated by the ideas of [8, 10, 1, 9,16, 17] and the line search used in [11], this paper is to present hybrid conjugate gradient method for solving system of nonlinear equations using prominents Dai-Yaun and Hestens-stiefel conjugate gradient parameters.

Consequently, this article is organized as follows: Next section is the details of our new method. Some numerical results will reported in Section 3. Finally the conclusion and result discussion.

2. DETAILS OF THE PROPOSED METHOD

This section gives the details of our proposed hybrid using the two well-known conjugate gradient methods (i.e. Dai-Yaun and Hestens-stiefel). The classical conjugate gradient direction is given by

$$d_{k+1} = -F_{k+1} + \beta_k d_k \tag{3}$$

Where β_k is termed as conjugate gradient parameter. Recall that, Dai-Yaun conjugate gradient parameter is defined by

$$\beta_k^{DY} = \frac{F_{k+1}^T F_{K+1}}{d_k^T y_k} \tag{4}$$

And Hestens-stiefel conjugate gradient parameter is defined as

$$\beta_k^{HS} = \frac{F_{K+1}^T y_k}{d_k^T y_k} \tag{5}$$

Let $\boldsymbol{\eta}$ belong to $[0,1]$

Suppose that

$$\beta_k^* = \boldsymbol{\eta}\beta_k^{DY} \tag{6}$$

$$\beta_k^{**} = \boldsymbol{\eta}\beta_k^{HS} + (1-\boldsymbol{\eta})\beta_k^{DY} \tag{7}$$

$$d_{k+1} = -F_{k+1} + \boldsymbol{\eta}\boldsymbol{\beta}_{\boldsymbol{k}}^{\boldsymbol{HS}} d_k \tag{8}$$

$$d_{k+1} = -\ F_{k+1} + [\eta\beta_k^{HS} + (1-\eta)\beta_K^{DY}]d_k \tag{9}$$

Equating (11) and (12) we have

$$\boldsymbol{\eta}\beta_k^{DY} = \boldsymbol{\eta}\beta_k^{HS} + (1-\eta)\beta_k^{DY} \tag{10}$$

After some algebra we get

$$\boldsymbol{\eta} = \frac{\beta_K^{DY}}{2\beta_K^{DY} - \beta_K^{HS}} \tag{11}$$

Substituting $\boldsymbol{\eta}$ in to equation (10) we have

$$\beta_k^{**}= \frac{\beta_k^{DY}\beta_k^{HS}}{2\beta_k^{DY}-\beta_k^{HS}} + (1- \frac{\beta_k^{DY}}{2\beta_k^{DY}-\beta_k^{HS}})\beta_k^{DY} \tag{12}$$

After some algebra of above equation, we get new formula denoted by β_k^{**} is defined by

$$\beta_k^{**} = \frac{2\left(\|F_{k+1}\|^2\right)^2}{d_k^T y_k \,(2\|F_{k+1}\|^2 - F_{k+1}^T y_k d_k^T y_k)}$$

However, we choose the steplength α_k, we proposed to use the well-known gradient line search defined by

$$\|F(x_k + \alpha_k d_k)\| \leq \delta_k \|F_k\|, \tag{13}$$

for $\propto_{k+1}= \frac{\propto_k}{2}$

Therefore, we defined our iterative scheme as:

Let $z_k = x_k + d_k$, $k = 0,1,2,\dots$, then the hyperplane

$$H_k = \{x \in R^n \mid (x - z_k)^T F(z_k) = 0\} \tag{14}$$

strictly separates x_k from the solution set of (1). Therefore, from this facts, Solodov and Svaiter[8] advised to let the next iterate x_{k+1} be the projection of x_k onto this hyperplane H_k. Therefore, x_{k+1} is now defined as

$$x_{k+1} = x_k - \frac{F(z_k)^T (x_k - z_k)}{\| F(z_k)\|^2} F(z_k). \tag{15}$$

Henrce our proposed direction d_k is given by

$$d_k = -F(x_0) \quad ,k = 0 \quad and \quad d_k = -F(x_k) + \beta_k^{**} d_k, \quad k \geq 1 \tag{16}$$

Algorithm

Step1: Given x_0, δ>0, and $\alpha_0 > 0$, then compute $d_0 = -F(x_0)$ and set $k = 0$.

Step 2: Test a stopping criterion. If yes, then stop; otherwise, continue with Step 3.

Step 3: Compute α_k by the line search (13).

Step 4: Compute $x_{k+1} = x_k - \frac{F(z_k)^T (x_k - z_k)}{\| F(z_k) \|^2} F(z_k)$.

Step 5: Compute the search direction by (16).

Step 5: Consider $k = k+1$ and go to step 2.

3. NUMERICAL RESULTS

In this section, we compare the performance of our conjugate gradient method for solving a a system of nonlinear equation with new proposed inexact PRP conjugate gradient method for solving symmetric nonlinear equation [15]. The both codes were written using Matlab8.0 R2012b and run on personal computer 2.10 GHz CPU processor and 4 GB Ram memory with the following initial parameters as:

Algorithm: $\propto_k = 1$ and $\delta_k = 0.9$

INPRP: we set $\omega_1 = w_2 = 10^{-4}$, $\alpha_1 = 0.01$, $r = 0.2$, and $\eta = {}^{1}/_{(k+1)^2}$

TABLE1: Numerical comparison between our algorithm and Inexact PRP method

Algorithm						Inexact PRP		
Problem	x_0	n	Iteration	β_k^{**} Time(s)	$\|\|F_k\|\|$	Iteration	INPRP Time(s)	$\|\|F_k\|\|$

1	0.5e	100	15	0.002951	9.4630e-05	40	0.012713	9.8002e-05
		200	16	0.009085	5.4211e-05	42	0.022958	8.4489e-05
		300	16	0.005871	6.6395e-05	43	0.027492	8.0793e-05
		400	16	0.027642	7.6666e-05	43	0.029832	9.3292e-05
		500	16	0.014245	8.1438e-05	44	0.032500	8.5715e-05
		600	16	0.017171	8.9211e-05	44	0.032020	9.3897e-05
2	0.5e	100	18	0.009311	3.6023e-05	50	0.017604	8.5435e-05
		150	18	0.030552	4.4107e-05	51	0.087215	8.1697e-05
		350	18	0.012761	6.7375e-05	52	0.049630	9.7437e-05
		400	18	0.008376	7.2026e-05	53	0.045349	8.1329e-05
		600	18	0.010549	8.8214e-05	53	0.067852	9.9607e-05
		800	19	0.030211	8.8875e-05	54	0.060670	8.9802e-05
3	0.5e	200	18	0.014740	6.7182e-05	46	0.026009	9.878e-05
		50	17	0.018771	6.0389e-05	44	0.025296	8.0968e-05
		5000	20	0.243431	7.5107e-05	53	0.327190	8.7308e-05
		2500	21	0.966711	6.5632e-05	56	1.071846	9.2922e-05
		0	19	0.038222	5.3343e-05	49	0.051142	9.3974e-05
		800	19	0.016911	7.3043e-05	51	0.094136	7.8444e-05
		1500	19	0.058713	5.9640e-05	50	0.083763	8.2033e-05

4	0.5e	300	18	0.020292	8.2211e-05	47	0.059375	9.4399e-05
		1500	19	0.042501	7.3043e-05	51	0.106967	7.8444e-05
		90	19	0.011787	4.9898e-05	49	0.042068	8.7904e-05
		850	17	0.005093	8.1020e-05	45	0.031743	8.4816e-05
		120	19	0.222768	5.4985e-05	49	0.025648	9.6866e-05
		50	17	0.005209	9.3554e-05	45	0.010681	9.7937e-05
			17	0.005158	6.0389e-05	44	0.014047	8.0968e-05
5	0.1e	500	13	0.013564	5.0425e-05	80	0.120601	8.3848e-05
		300	12	0.012389	8.1186e-05	78	0.097641	9.9133e-05
		5500	14	0.073117	6.5895e-05	90	0.816910	9.1120e-05
		50						
			12	0.007587	4.0471e-05	70	0.053775	8.0932e-05
6	0.3e	600	10	0.011121	1.7721e-05	26	0.361364	8.2130e-05
		300	10	0.004294	2.1211e-05	24	0.025821	7.9039e-05
		1000	10	0.008536	5.3013e-05	27	0.034039	8.9196e-05
		500	10	0.011177	1.0963e-05	25	0.014294	9.2550e-05
		1200	10	0.016973	7.2726e-05	27	0.031654	9.9550e-05
8	0.4e	700	3	0.006582	1.7430e-05	18	0.022914	2.1365e-05
		200	3	0.003778	5.8411e-05	17	0.019357	9.3168e-05

		900	3	0.005949	1.97644-05	18	0.030703	2.4225e-05
		80	3	0.003450	3.6942e-05	17	0.010260	5.8925e-05
		1000	3	0.005858	2.0833e-05	18	0.017495	2.5535e-05
12	0.15e	800	19	0.377682	6.9886e-05	54	0.0105131	9.5440e-05
		80	17	0.024725	7.1025e-05	50	0.074375	8.1633e-05
		500	19	0.031424	5.5250e-05	53	0.074794	9.6762e-05
		1500	19	0.06682	7.9463e-05	56	0.210276	9.5695e-05
14	0.6e	1000	3	0.007782	7.7506e-07	20	0.041203	9.0632e-05
		1500	3	0.040275	5.8569e-05	21	0.398142	9.4995e-07
		2500	3	0.017596	1.2264e-06	21	0.502070	7.5612e-05
		10	3	0.002599	7.7563e-08	17	0.032009	9.3873e-05
		100	3	0.002446	2.4528e-07	19	0.013404	6.7025e-05
15	0.9e	1000	21	0.148988	2.0303e-05	49	0.201235	9.4052e-05
		400	20	0.063128	8.0306e-05	48	0.078092	9.7229e-05
		900	21	0.030693	1.9261e-05	49	0.196178	8.9226e-05
		1800	21	0.167460	2.7239e-05	50	0.325544	9.3467e-05
		4500	21	0.462552	4.3069e-05	52	0.674398	8.1083e-05
16	0.4e	1100	8	0.035858	2.6993e-07	34	0.083381	9.3043e-05
		500	8	0.025612	1.8199e-05	34	0.036574	6.2730e-05
		50	6	0.015257	6.1429e-05	32	0.027877	6.7225e-05

		1500	8	0.073276	3.1522e-07	35	0.117161	6.1686e-05
		8100	8	0.333290	7.3251e-07	36	0.518012	8.1536e-05

Problem 1:

$$F(x_i) = e^{x_i} - 1, \qquad i = 1,2,\dots,n.$$

Problem2

$$F(x_i) = x_i^2 - 4 \qquad i = 1,2,\dots,n.$$

Problem3:

$$F(x_i) = x_i x_{i+1} - 1, \qquad i = 1,2,\dots,n-1.$$

$$F(x_n) = x_n x_1 - 1,$$

Problem4:

$$F(x_i) = x_i^2 - 1, \qquad i = 1,2,\dots,n.$$

Problem5:

$$F(x_i) = x_i^2 + x_i - 2, \qquad i = 1,2,\dots n.$$

Problem6:

$$F(x_i) = \cos(x_{i-1}) + x_i - 1, \;\; i = 1,2,\dots,n.$$

Problem8:

$$F(x_i) = x_i^2 - \cos(x_i) - 1, \;\; i = 1,2,\dots,n.$$

Problem12:

$$F(x_i) = x_i - 3x_i(\sin(x_i)/3) - 0.66 + 2 \;\; i = 1,2,..,n.$$

Problem14:

$$F(x_i) = e^{x_i^2} - 1 - \cos(1 - x_i) \quad i = 1,2,..,n.$$

Problem15:

$$F(x_i) = x_i^2 + x_i - 3\log(x_{i+3}) - 9, \qquad i = 1,2,\dots,n.$$

Problem16:

$$F(x_i) = x_{i-1}^2 0.2 - 2 \quad i = 1,2,\dots,n.$$

4. CONCLUSION

This paper developed a hybrid method for solving system of algebraic nonlinear equations. The proposed scheme was entirely derivative-free and matrix-free iterative approach which possesed less number of iteration and CPU time in second. However, our algorithm doesn't required computation and storing of $n \times n$ Jacobian matrix. Numerical comparison with original inexact PRP method for solving symmetric nonlinear equations shows or scheme is promising. Furthermore, the numerical experiment on benchmark test problems shows that the proposed method large-scale system of nonlinear equations with minimun CPU time and number of iterations compared to the existing algorithms (IPRP).

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