A-Stable Block ETRs/ETR2s Methods for Stiff ODEs

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***Abstract*- We present a class of fourth and sixth order A-stable block extended trapezoidal rule of first kind (ETRs) and extended trapezoidal rule of second kind (ETR2s) methods which are found to be adequate for the numerical integration of stiff ordinary differential equations. The single continuous formulation of this methods are evaluated at some grid and interior points yielding the multi-discrete schemes which are implemented in block form thereby generating simultaneously approximate solutions $y_1, y_2, \ldots, y_k$ at once without recourse to predictors. By this approach, the need for starters is eliminated. The stability properties of the block ETRs/ETR2s methods discussed and were shown to preserve the A-stability property of the trapezoidal rule, their absolute stability regions also presented. The newly derived block methods were implemented on five stiff systems of ordinary differential equations occurring in real life to show efficiency and accuracy.**

***Keywords*: Multi-Step Collocation Method, Extended Trapezoidal Rule, Interpolation and Collocation, ETRs/ ETR2s.**

I. INTRODUCTION

A great many physical occurrences give rise to problems that often result in ODEs. When we solve a differential equation, we are in effect solving the physical problems it represents. Traditionally, solution to differential equations was derived using analytical methods. These solutions are often useful as they provide excellent insight into the behavior of some systems. However, certain differential equations are very difficult to solve by any means other than an approximate solution by the application of numerical methods [3]. The common numerical methods used to solve ODEs are the one-step (multistage) method like the Euler and trapezoidal methods and multistep methods [18]. Most physical problems modeled in kinetics, chemical reactions, process control and electrical circuit theory often result to stiff ODEs where processes with wide varying time constrains are usually encountered. It should be recalled that stiff initial value problem were first encountered in the study of motion of spring of varying stiffness, from which the problem derive its name [8]. Extended Trapezoidal Rule of First kind and that of Second Kind were introduced by [4]. According to the authors, the method(s) belong to the group of symmetric Schemes and are categorized into the family of Boundary Value Methods (BVMs). Extended trapezoidal rules have been extensively studied by the following scholars [10, 11 and 19].

A. *Symmetric schemes*

We group as symmetric schemes BVMs having the following general properties:

1. They have an odd number of steps, $k = 2v - 1,\ v \geq 1,$ and must be used with $(v, v-1)$ –boundary conditions (that is, they require $v-1$ initial and $v-1$ final additional methods).
2. The corresponding polynomials $\rho(z)$ have skew-symmetric coefficients. That is

$z^k \rho(z^{-1}) = -\rho(z)$;

3. The corresponding polynomials $\sigma(z)$ have symmetric coefficients. That is

$z^k \sigma(z^{-1}) = \sigma(z)$

4. $D_{v,v-1} \equiv \subset^{-}$

Note: Multi-step collocation approach which we have adopted in this paper allows for flexibility of obtaining all our discrete schemes from the same continuous formulation of the main method, thereby eliminating the need for starters and its implementation to ODEs also eliminates any boundary conditions (that is, the $v-1$ initial and $v-1$ final additional methods stated above). However, it is easy to show that our methods are symmetric.

Remark: Show that the sixth order extended trapezoidal rule of first kind is symmetric.

Proof

From (21), we have that

$$\alpha_2 = -1,\ \alpha_3 = 1,\ \beta_0 = 11, \beta_1 = -93, \beta_2 = 802, \beta_3 = 802, \beta_4 = -93 \text{ and } \beta_5 = 11$$

i. Consider the LHS of (2) in subsection A , we obtained

$$\rho(z^{-1}) = z^{-3} - z^{-2}$$

Then, $z^5 \rho(z^{-1}) = z^5 (z^{-3} - z^{-2}) = z^2 - z^3$

Now, RHS of (2), yields

$$-\rho(z) = -\left(z^3 - z^2\right) = z^2 - z^3$$

ii. From the LHS of (3), we have that

$$\sigma(z^{-1}) = 11z^{-5} - 93z^{-4} + 802z^{-3} + 802z^{-2} - 93z^{-1} + 11$$

$$z^5 \sigma(z^{-1}) = z^5 \left(11z^{-5} - 93z^{-4} + 802z^{-3} + 802z^{-2} - 93z^{-1} + 11\right)$$

$$= 11 - 93z + 802z^2 + 802z^3 - 93z^4 + 11z^5$$

From RHS of (3), yields

$\sigma(z) = 11z^5 - 93z^4 + 802z^3 + 802z^2 - 93z + 11$. In both cases, RHS=LHS. QED

In recent years, the problem of deriving more advanced and efficient numerical methods for stiff problems has received a great deal of attention and as a result, a wide variety of approaches have been proposed. A potentially good numerical method for the solution of stiff system of ODEs of the form

$$\left.\begin{aligned} y'(x) &= f(x, y(x)) \\ y(a) &= \eta, a \le x \le b \end{aligned}\right\} \quad (1)$$

must have good accuracy and some reasonably wide region of absolute stability [6, 14]. One of the first and most important stability requirements particularly for linear multistep method is A-stability which was proposed in [7, 14]. However, the requirement of A-stability put some limitations on the choice of suitable LMMs. Dahlquist proved that the order of an A-stable LMM must be greater or equal to two (2) and that an A-stable multistep method must be implicit.

The general k-step method for (1) given by [13] is written in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j},\ \alpha_k \neq 0 \tag{2}$$

where,

α_j and β_j are coefficients of the method to be uniquely determined, h a constant step size and k the step number. We propose in this study a basis function of the form:

$$y(x) = \sum_{j=0}^{m} \varphi_j (x - x_k)^j \tag{3}$$

Equation (3) can now be used to generate the high order A-stable block ETR$_2$s methods.

B. *Convergence of Linear Multi-Step Methods*

Definition 1**:** A linear multistep method defined by a formula of the form (2) is said to be convergent in a region $[t_0, t_1]$ if

$$\lim_{h \to 0} x_h(t) = x(t),\ t \in [t_0, t_1] \tag{4}$$

provided only that

$$\lim_{h \to 0} x_h(t + jh) = x_0,\ 0 \le j \le k \tag{5}$$

Here $x_h(t)$ is the numerical solution computed using a step size of h and $x(t)$ is the theoretical solution. This definition is natural enough. The following definitions are not so natural, but nevertheless extremely useful.

*Definition 2***:** *Consider a linear multistep method corresponding to a relation of the form (2). Set*

$$P(\lambda) = \alpha_k \lambda^k + \alpha_{k-1}\lambda^{k-1} + \alpha_{k-2}\lambda^{k-2} + \ldots + \alpha_1\lambda + \alpha_0$$

$$Q(\lambda) = \beta_k \lambda^k + \beta_{k-1}\lambda^{k-1} + \beta_{k-2}\lambda^{k-2} + \ldots + \beta_1\lambda + \beta_0$$

We shall say that the method (2) is stable if the roots of the polynomial $P(\lambda)$ lie in disk $|\lambda| \le 1$ and if each root such that is $|\lambda| = 1$ simple. The method (2) is said to be consistent if $P(1) = 0$ and $P'(1) = Q(1)$

Theorem 1: Consider a linear multistep method corresponding to a relation of the form (2). Then this method is convergent if and only if it is both consistent and zero stable.

C: *The Order of Linear Multi-Step Methods***.**

The order of a linear multi-step method is an integer that corresponds to the number of terms in the Taylor expansion of the solution of the solution that a multi-step method stimulates. Let us represent the linear multi-step method (2) as a linear functional

$$L[x] = \sum_{j=0}^{k} \left[\alpha_j x(jh) - h\beta_j f(jh)\right]$$

$$= \sum_{j=0}^{k} \left[\alpha_j x(jh) - h\beta_j x'(jh)\right]$$

Here we let $k = n$ to simplify our notational and assume that the first value of equation (2) begins $t = 0$ at rather, than $t = (n-k)h$. Now

$$x(jh) = \sum_{i=0}^{\infty} \frac{(jh)^i}{i!} x^{(i)}(0)$$

$$x'(jh) = \sum_{i=0}^{\infty} \frac{(jh)^i}{i!} x^{(i+1)}(0)$$

and so we can write

$$L[x] = \sum_{j=0}^{k} \left[\alpha_j \sum_{i=0}^{\infty} \frac{(jh)^i}{i!} x^{(i)}(0) - h\beta_j \sum_{i=0}^{\infty} \frac{(jh)^i}{i!} x^{(i+1)}(0)\right] \tag{6}$$

Collecting terms proportional to $x(0)$ and $x'(0),\ldots$ (or equivalently by their degree in h) we have

$$L[x] = u_0 x(0) + u_1 h x'(0) + u_2 h^2 x''(0) + \ldots$$

where

$$u_0 = \sum_{i=0}^{k} \alpha_i$$

$$u_1 = \sum_{i=0}^{k} (i\alpha_i - \beta_i)$$

$$u_2 = \sum_{i=0}^{k} (\frac{1}{2} i^2 \alpha_i - i\beta_i)$$

$$.u_j = \sum_{i=0}^{k} (\frac{i^j}{j!} \alpha_i - \frac{i^{j-1}}{(j-1)!} \beta_i) \tag{7}$$

Theorem 2: The following three properties of the LMM (2) are equivalent:

1. $u_0 = u_1 = u_2 = ... = u_m = 0$
2. $L[P] = 0$ for each polynomial P of degree $\leq m$
3. $L[x]$ is $0(h^{m+1})$ for all $x \in C^{m+1}$

Proof

- (1) $\Rightarrow$ (2):
  If (1) is true then
  $$L[x] = 0 + 0 + ... + 0 + u_{m+1} h^{m+1} x^{(m+1)}(0) + u_{m+2} h^{m+2} x^{(m+2)}(0) + ...$$

  But if P is a polynomial of degree $\leq m$ then $P^{(m+1)}(x) = 0.$ therefore, for such a polynomial

  $$L[x] = 0 + 0 + ... + 0 + u_{m+1} h^{m+1} P^{(m+1)}(0) + u_{m+2} h^{m+2} P^{(m+2)}(0) + ...$$

  $$= 0$$

- (2) $\Rightarrow$ (3):
  If $x \in C^{m+1}$, then by Taylor's Theorem we can write

  $$x(t) = P(t) + R(t)$$

where $P(t)$ is a polynomial of degree m and

$$R(t) = \frac{x^{(m+1)}(\xi)}{(m+1)!} l^{m+1}$$

Notice that

$$\left. \frac{d^j R}{dt^j} \right|_{t=0} = 0$$

Hence

$$L[x] = L[P] + L[R]$$

$$L[x] = 0 + u_{m+1} h^{m+1} x^{(m+1)}(0) + u_{m+2} h^{m+2} x^{(m+2)}(0) + ...$$

$$L[x]=0(h^{m+1})$$

- (3) $\Rightarrow$ (1)
  If (3) is true, then we must have $u_0=u_1=u_2=\ldots=u_m=0.$ Henc, (3) implies (1).

Definition 3: The order of a LMM is the unique natural number m such that

$$0=u_0=u_1=u_2=\ldots=u_m\neq u_{m+1}$$

II. METHODOLOGY

Case 1: *Derivation of fourth order Extended Trapezoidal Rule of first kind*

Equation (3) can be reformulated as a polynomial function:

$$\left.\begin{aligned} &Y(x)=\sum_{j=0}^{m}\varphi_j(x-x_k)^j\equiv y(x),\\ &x_k\le x\le x_{k+p}\end{aligned}\right\} \tag{8}$$

Over each of the sub-interval (x_k, x_{k+p}) of (a,b) where, m is appropriately chosen. This shall be used as basis function to derive the LMM in continuous form.

The technique which is being employed is using the trial or basis function

$$Y(x)=\sum_{j=0}^{n+1}\varphi_j(x-x_k)^j\equiv y(x),\ x_k\le x\le x_{k+p} \tag{9}$$

This satisfies the unperturbed ODE:

$$\left.\begin{aligned} &Y'(x)=f(x,y(x)),\ x_k\le x\le x_{k+p}\\ &Y(x_k)=Y_k\end{aligned}\right\} \tag{10}$$

Collocating equation (10) at $(n+1)$ points $x_{k+j},\ j=0,1,\ldots,\ n$ and interpolating the trial polynomial (9) at $x_{k+j},\ j=1,2,\ldots,\ n-1$ to give the require $(n+2)$ equations for the unique determination of φ_j.

Doing this, we write

$$\left.\begin{aligned} &f(x_{k+j})=f_{k+j},\ j=0,1,\ldots,\ n\\ &Y(x_{k+j})=Y_{k+j},\ j=1,2,\ldots,\ n-1\end{aligned}\right\} \tag{11}$$

To derive the fourth order block ETR$_2$s method, we set, $n=3$ in the equation (9), so that

$$Y(x)=\varphi_0+\varphi_1(x-x_k)+\varphi_2(x-x_k)^2+...+\varphi_8(x-x_k)^4 \qquad (12)$$

From equation (11), we have

$$\left.\begin{array}{l} Y'(x_k)=f_k \\ . \\ . \\ . \\ Y'(x_{k+3})=f_{k+3} \\ Y(x_{k+1})=Y_{k+1} \\ Y(x_{k+2})=Y_{k+2} \end{array}\right\} \qquad (13)$$

Using equation (12) in (13), we obtain the following equations

$$Y(x_{k+1})=\varphi_0+\varphi_1(\mu)+\varphi_2(\mu)^2+\varphi_3(\mu)^3+\varphi_4(\mu)^4=Y_{k+1}$$

$$Y'(x_k)=\varphi_1=f_k$$

$$Y'(x_{k+1})=\varphi_1+2\varphi_2(\mu)+3\varphi_3(\mu)^2+4\varphi_4(\mu)^3=f_{k+1}$$

$$Y'(x_{k+2})=\varphi_1+4\varphi_2(\mu)+12\varphi_3(\mu)^2+32\varphi_4(\mu)^3=f_{k+2}$$

$$Y'(x_{k+3})=\varphi_1+6\varphi_2(\mu)+27\varphi_3(\mu)^2+108\varphi_4(\mu)^3=f_{k+3}$$

where,

$\mu=x-x_k$

Representing this in matrix form yield

$$\begin{bmatrix} 1 & \mu & \mu^2 & \mu^3 & \mu^4 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & \mu & 2\mu^2 & 3\mu^3 & 4\mu^4 \\ 0 & \mu & 4\mu^2 & 12\mu^3 & 32\mu^4 \\ 0 & \mu & 6\mu^2 & 27\mu^3 & 108\mu^4 \end{bmatrix}\begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}=\begin{bmatrix} Y_{k+1} \\ f_k \\ f_{k+1} \\ f_{k+2} \\ f_{k+3} \end{bmatrix}$$

Solving for $\varphi_i, i = 1,2,..., n+1$ and substituting in equation (12) we obtained

$$\left.\begin{aligned} Y(x) = &\left(Y_{k+1} - \frac{3}{8} f_k - \frac{19}{24} f_{k+1} + \frac{5}{24} f_{k+2} - \frac{1}{24} f_{k+3}\right) + \left(\frac{1}{h} f_k\right)\mu + \left(\frac{3}{2h^2} f_{k+1} - \frac{3}{4h^2} f_{k+2} + \frac{1}{6h^2} f_{k+3} - \frac{11}{12h^2} f_k\right)\mu^2 \\ &+ \left(\frac{2}{3h^3} f_{k+2} - \frac{5}{6h^3} f_{k+1} - \frac{1}{6h^3} f_{k+3} + \frac{1}{3h^3} f_k\right)\mu^3 + \left(\frac{1}{8h^4} f_{k+1} - \frac{1}{8h^4} f_{k+2} + \frac{1}{24h^4} f_{k+3} - \frac{1}{24h^4} f_k\right)\mu^4 \end{aligned}\right\} \quad (14)$$

On evaluation of (14) at some end points and interior points, we obtained the following equations whose coefficients are presented in table 1 and 2.

$$y_{n+2} - y_{n+1} = \frac{h}{24}\left[- f_{n+3} + 13 f_{n+2} + 13 f_{n+1} - f_n\right]$$

$$y_{n+1} - y_n = \frac{h}{24}\left[f_{n+3} - 5 f_{n+2} + 19 f_{n+1} + 9 f_n\right]$$

$$y_{n+3} - y_{n+1} = \frac{h}{3}\left[f_{n+3} + 4 f_{n+2} + f_{n+1}\right] \quad (15)$$

Case 2: Derivation of fourth order Extended Trapezoidal Rule of Second kind

Collocating equation (10) at $(n-1)$ points $x_{k+j},\ j = 1,2,..., n-1$ and interpolating the trial polynomial (9) at $x_{k+j},\ j = 0,1,..., n-1$ to give the require $(n+2)$ equations for the unique determination of φ_j.

Doing this, we write

$$\left.\begin{aligned} f(x_{k+j}) &= f_{k+j},\ j = 1,2,..., n-1 \\ Y(x_{k+j}) &= Y_{k+j},\ j = 0,1,.., n-1 \end{aligned}\right\} \quad (16)$$

To derive the fourth order block ETR$_2$s method, we set, $n = 3$ in the equation (9), so that

$$Y(x) = \varphi_0 + \varphi_1(x - x_k) + \varphi_2(x - x_k)^2 + ... + \varphi_8(x - x_k)^4 \quad (17)$$

From equation (11), we have

$$\left.\begin{array}{l} Y'(x_{k+1}) = f_{k+1} \\ Y'(x_{k+2}) = f_{k+2} \\ Y(x_k) = Y_k \\ . \\ . \\ . \\ Y(x_{k+3}) = Y_{k+3} \end{array}\right\} \quad (18)$$

Using equation (12) in (13), we obtain the following equations

$$Y(x_k) = \varphi_0 = Y_k$$

$$Y(x_{k+1}) = \varphi_0 + \varphi_1(\mu) + \varphi_2(\mu)^2 + \varphi_3(\mu)^3 + \varphi_4(\mu)^4 = Y_{k+1}$$

$$Y(x_{k+2}) = \varphi_0 + 2\varphi_1(\mu) + 4\varphi_2(\mu)^2 + 8\varphi_3(\mu)^3 + 16\varphi_4(\mu)^4 = Y_{k+2}$$

and

$$Y'(x_{k+1}) = \varphi_1 + 2\varphi_2(\mu) + 3\varphi_3(\mu)^2 + 4\varphi_4(\mu)^3 = f_{k+1}$$

$$Y'(x_{k+2}) = \varphi_1 + 4\varphi_2(\mu) + 12\varphi_3(\mu)^2 + 32\varphi_4(\mu)^3 = f_{k+2}$$

where,

$$\mu = x - x_k$$

Representing this in matrix form yield

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \mu & \mu^2 & \mu^3 & \mu^4 \\ 1 & 2\mu & 4\mu^2 & 8\mu^3 & 16\mu^4 \\ 0 & \mu & 2\mu^2 & 3\mu^3 & 4\mu^4 \\ 0 & \mu & 4\mu^2 & 12\mu^3 & 32\mu^4 \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} Y_k \\ Y_{k+1} \\ Y_{k+2} \\ f_{k+1} \\ f_{k+2} \end{bmatrix}$$

Solving for $\varphi_i, i = 1,2,..., n+1$ and substituting in equation (12) we obtained

$$\left.\begin{aligned}Y(x)=Y_K+\left(\frac{3}{h}Y_{k+2}-\frac{4}{h}f_{k+1}-\frac{1}{h}f_{k+2}-\frac{3}{h}Y_k\right)\mu+\left(\frac{4}{h^2}Y_{k+1}-\frac{29}{4h^2}Y_{k+2}+\frac{8}{h^2}f_{k+1}+\frac{5}{2h^2}f_{k+2}+\frac{13}{4h^2}Y_k\right)\mu^2\\+\left(\frac{11}{2h^3}Y_{k+2}-\frac{4}{h^3}Y_{k+1}-\frac{5}{h^3}f_{k+1}-\frac{2}{h^3}f_{k+2}-\frac{3}{2h^3}Y_k\right)\mu^3+\left(\frac{1}{h^4}Y_{k+1}-\frac{5}{4h^4}Y_{k+2}+\frac{1}{h^4}f_{k+1}+\frac{1}{2h^4}f_{k+2}+\frac{1}{4h^4}Y_k\right)\mu^4\end{aligned}\right\} \tag{19}$$

On evaluation of (19) at some end and interior points, we obtained the following equations whose coefficients are presented in table 1 and 2.

$$\begin{aligned}&\frac{1}{12}\left(y_{n+3}+9y_{n+2}-9y_{n+1}-y_n\right)=\frac{h}{2}\left[f_{n+2}+f_{n+1}\right]\\&\frac{1}{30}\left(27y_{n+2}-24y_{n+1}-3y_n\right)=\frac{h}{30}\left[-f_{n+3}+14f_{n+2}+17f_{n+1}\right]\\&\frac{1}{2}\left(y_{n+2}-y_n\right)=\frac{h}{6}\left[f_{n+2}+4f_{n+1}+f_n\right]\end{aligned} \tag{20}$$

Following a similar procedure as in case 1 and 2, we obtained ETRs and ETR$_2$s of up to order 6.

Case 3: *Sixth order Extended Trapezoidal Rule of first kind*

$$\begin{aligned}y_{n+3}-y_{n+2}&=\frac{h}{1440}\left[11f_{n+5}-93f_{n+4}+802f_{n+3}+802f_{n+2}-93f_{n+1}+11f_n\right]\\y_{n+2}-y_n&=\frac{h}{90}\left[f_{n+5}-6f_{n+4}+14f_{n+3}+14f_{n+2}+129f_{n+1}+28f_n\right]\\y_{n+2}-y_{n+1}&=\frac{h}{1440}\left[-11f_{n+5}+77f_{n+4}-258f_{n+3}+1022f_{n+2}+637f_{n+1}-27f_n\right]\\y_{n+4}-y_{n+2}&=\frac{h}{90}\left[-f_{n+5}+34f_{n+4}+114f_{n+3}+34f_{n+2}-f_{n+1}\right]\\y_{n+5}-y_{n+2}&=\frac{h}{160}\left[51f_{n+5}+219f_{n+4}+114f_{n+3}+114f_{n+2}-21f_{n+1}+3f_n\right]\end{aligned} \tag{21}$$

and

Case 4: *Sixth order Extended Trapezoidal Rule of Second kind*

$$\frac{1}{120}(-y_{n+5}+15y_{n+4}+80y_{n+3}-80y_{n+2}-15y_{n+1}+y_n)=\frac{h}{2}\left[f_{n+3}+f_{n+2}\right]$$

$$\frac{1}{24}\left(381y_{n+4}+2692y_{n+3}-2592y_{n+2}-516y_{n+1}+35y_{n}\right)=\frac{h}{2}\left(f_{n+5}+157f_{n+3}+167f_{n+2}\right)$$

$$\frac{1}{24}\left(43y_{n+4}+80y_{n+3}-108y_{n+2}-16y_{n+1}+y_{n}\right)=\frac{h}{2}\left(f_{n+4}+8f_{n+3}+6f_{n+2}\right)$$

$$\frac{1}{24}\left(y_{n+4}+28y_{n+3}-28y_{n+2}-y_{n}\right)=\frac{h}{2}\left(f_{n+3}+3f_{n+2}+f_{n+1}\right)$$

$$\frac{1}{24}\left(9y_{n+4}+208y_{n+3}-108y_{n+2}-144y_{n+1}+35y_{n}\right)=\frac{h}{2}\left(8f_{n+3}+18f_{n+2}-f_{n}\right) \tag{22}$$

III. ANALYSIS OF BASIC PROPERTIES OF THE METHODS

Consider equation (2), writing this in block form we have

$$A^{(0)}Y_m = A^{(1)}y_n + hdf(y_n) + hbF(Y_m) \tag{23}$$

where,

$$Y_m = \left[y_{i+1}\right]^T, y_n = \left[y_{i-2}\right]^T, F(Y_m) = \left[f_{i+1}\right]^T, f(y_n) = \left[f_{i-2}\right]^T, i = n(n+1)n+2$$

The linear operator $L\{y(x);h\}$ associated with the block (15) can be defined as:

$$L\{y(x);h\} = A^{(0)}Y_m - A^{(1)}y_n - hdf(y_n) - hbF(Y_m) \tag{24}$$

where, $y(x_n)$ is any sufficiently differentiable vector valued function. By Taylor series expansion, we have that

$$L\{y(x);h\} = c_{p+1}h^{(p+1)}y^{(p+1)}(x_n) + 0(h^{p+2}), x \in \left[x_n, x_{n+1}\right] \tag{25}$$

here, c is regarded as the error constants, p the order of (2).

A. *Order of the fourth order block ETRs*

Applying (23) on (15), we obtained

$$L\{y(x);h\}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&1&0\end{bmatrix}\begin{bmatrix}y_{n}\\y_{n+1}\\y_{n+2}\end{bmatrix}-\begin{bmatrix}-\frac{9}{24}&-\frac{19}{24}&\frac{5}{24}&-\frac{1}{24}\\-\frac{1}{24}&\frac{13}{24}&\frac{13}{24}&-\frac{1}{24}\\0&\frac{1}{3}&\frac{4}{3}&\frac{1}{3}\end{bmatrix}\begin{bmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix} \tag{26}$$

Expanding (26) in Taylor Series gives

$$\left.\begin{matrix}\sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j-y_n+\frac{9h}{24}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{-\frac{19}{24}(1)^j+\frac{5}{24}(2)^j-\frac{1}{24}(3)^j\right\}\\ \sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^j-y_n^{j+1}+\frac{h}{24}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{13}{24}(1)^j+\frac{13}{24}(2)^j-\frac{1}{24}(3)^j\right\}\\ \sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j-y_n^{j+1}-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{1}{3}(1)^j+\frac{4}{3}(2)^j-\frac{1}{3}(3)^j\right\}\end{matrix}\right\}=\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{27}$$

Following (25) and definition (3), we tabulate in tables 4 and 5 the order and error constants of the method.

B. *Order of the fourth order block ETR_2s*

Similarly, applying (23) on (20) yields;

$$L\{y(x);h\}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}-\begin{bmatrix}1\\1\\1\end{bmatrix}\left[y_n\right]-\begin{bmatrix}\frac{3}{8}&\frac{19}{24}&-\frac{5}{24}&\frac{1}{24}\\\frac{1}{3}&\frac{4}{3}&\frac{1}{3}&0\\\frac{3}{8}&\frac{9}{8}&\frac{9}{8}&\frac{3}{8}\end{bmatrix}\begin{bmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix} \tag{28}$$

Expanding (28) in Taylor Series yields

$$\left.\begin{array}{l}\sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j - y_n - \frac{3h}{8}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{19}{24}(1)^j - \frac{5}{24}(2)^j + \frac{1}{24}(3)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^j - y_n - \frac{h}{3}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{4}{3}(1)^j + \frac{1}{3}(2)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j - y_n - \frac{3h}{8}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{9}{8}(1)^j + \frac{9}{8}(2)^j + \frac{3}{8}(3)^j\right\}\end{array}\right\} = \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{29}$$

From (29), we obtained the order and error constants which is shown in tables 4 and 5.

C. *Zero-Stability of the methods*

Definition 4: A block method is said to be zero stable if as $h \to 0,$ the roots $r_j = 1(2)k$ of the first characteristics polynomials $\rho(\lambda)$ is given by

$$\rho(\lambda) = \det\left[\sum_{i=0}^{k} A^{(i)}\lambda^{k-i}\right] = 0 \tag{30}$$

satisfies $|\lambda_j| \le 1$, the multiplicity must not exceed two [12].

Following [2], we obtain the first characteristics polynomial of the *4th Order* ETRs as

$$\rho(\lambda) = \det\left[\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\right]$$

$$\rho(\lambda) = \det\begin{bmatrix}\lambda&0&-1\\0&\lambda&-1\\0&0&\lambda-1\end{bmatrix}$$

$$\rho(\lambda) = \lambda^2(\lambda-1) = 0$$

This implies that $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$. Hence by definition (2) and (4), the *4th Order* block ETRs (15) is zero-stable. Following theorem (1) and by [5, 8, 9 and 13], the block method is convergence since it is consistent and zero stable.

D. *Absolute Stability Regions of the Block ETRs and ETR$_2$s Methods*

Definition 5. A Numerical method is said to be A-stable, if its region of absolute stability contains the whole of the left hand complex plane.

The absolute stability regions of the **ETRs** and **ETR2s** methods are plotted (Fig. 1). Both methods were shown to be A-stable and therefore suitable for the solution stiff system of ordinary differential equations.

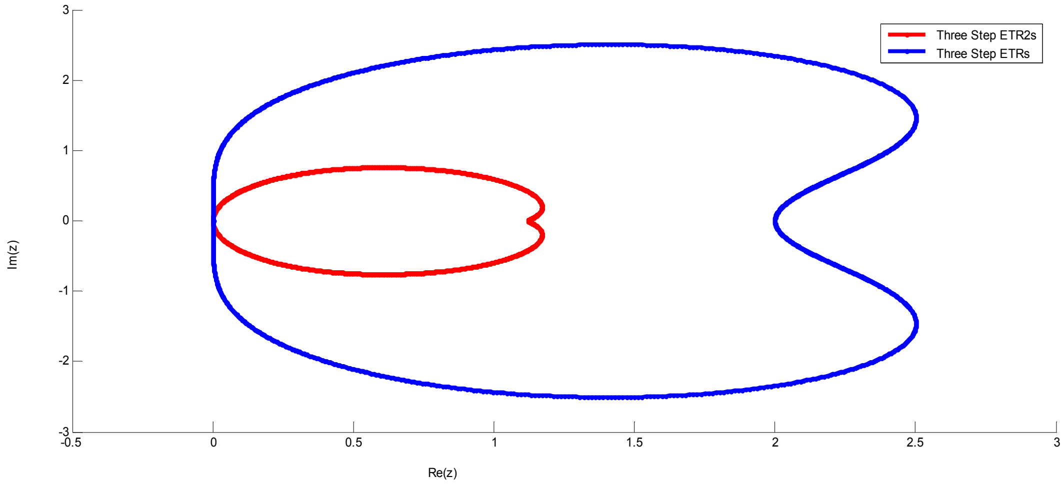

Figure 1. Regions of Absolute Stability of the *4th Order* block ETRs and ETR2s method.

IV. NUMERICAL EXPERIMENT

In order to assess the performance of our block methods, we consider five real life systems of first order ordinary differential equations.

Problem 1: A numerical example solved by [14].

$$y_1' = -\alpha y_1 - \beta y_2 + (\alpha + \beta - 1)e^{-x}$$

$$y_2' = \beta y_1 - \alpha y_2 - (\alpha - \beta - 1)e^{-x}$$

with initial value $y(0) = (1,1)^T$. In order to make this system homogeneous, we introduce an additional variable $y_3' = 1,\ y_3(0) = 0.$ the eigenvalues of the Jacobian associated with the resulting system are $-\alpha \pm i\beta,\ 0.$ this problem has theoretical solution as $y_1(x) = y_2(x) = e^{-x}$. Results are obtained when $\alpha = 1,\ \beta = 30$ and the value of h chosen was 0.09.

TABLE 1. Maximum Absolute Errors of Problem 4.1

$h = 0.09$ x	N	$y_i, i = 1,2$	SDMM [14] K=5	4th Order ETRs	4th Order ETR2s	6th Order ETRs	6th Order ETR2s
4.5	50	y_1	0.3e-11	5.2e-10	5.1e-10	1.6e-12	1.6e-12
		y_2	0.3e-11	2.2e-10	2.2e-10	2.2e-14	2.1e-14
9	100	y_1	0.3e-14	5.8e-12	5.7e-12	1.7e-14	1.6e-14
		y_2	0.3e-14	2.5e-12	2.5e-12	2.4e-16	2.3e-16
13.5	150	y_1	0.7e-16	6.4e-14	6.4e-14	2.0e-16	2.0e-16
		y_2	0.6e-16	2.3e-14	2.3e-14	2.7e-18	2.8e-18
18	200	y_1	0.1e-19	7.1e-16	7.1e-16	2.2e-18	2.2e-18
		y_2	0.2e-19	3.1e-16	3.0e-16	2.9e-20	2.8e-20

Problem 2**:** Considering the discharge valve on a 200 -gallon tank that is full of water opened at time $t = 0$ and 3 gallons per second flow out. At the same time 2 gallons per second of 1 percent chlorine mixture begin to enter the tank. Assume that the liquid is being stirred so that the concentration of chlorine is consistent throughout the tank. The task is to determine the concentration of chlorine when the tank is half full. It takes 100 seconds for this moment to occur, since we lose a gallon per second. If $y(t)$ is the amount of chlorine in the tank at time t, then the rate chlorine is entering is $\frac{2}{100}$ gal/sec and it is leaving at the rate $3\left[\frac{y}{200-t}\right]$ gal/sec.

Thus, the resulting IVP is

$$\frac{dy}{dt} = \frac{2}{100} - 3\frac{y}{200-t},\ y(0) = 0,\ 0 \le t \le 1,\ h = 0.1$$

whose theoretical solution is

$$y(t) = 2 - \frac{1}{100}t - 2\left[1 - \frac{5t}{1000}\right]^3$$

[See (Areo and Adeniyi, 2014)]

TABLE 2. Maximum of Absolute Errors for Problem 4.2

x	*Areo & Adeniyi [1] 6th Order*	*4th Order ETRs*	*6th Order ETRs*
0.1	0	1×10^{-17}	4×10^{-17}
0.2	0	0	0
0.3	2.40×10^{-11}	6×10^{-17}	2×10^{-17}
0.4	2.40×10^{-11}	3×10^{-17}	1×10^{-17}
0.5	2.40×10^{-11}	4×10^{-17}	1×10^{-17}
0.6	3×10^{-11}	0	1×10^{-16}
0.7	3×10^{-11}	0	1×10^{-16}
0.8	3×10^{-11}	0	2×10^{-16}
0.9	3×10^{-11}	7×10^{-16}	1×10^{-16}
1.0	3×10^{-11}	7×10^{-16}	0

Problem 3: As our third numerical example, we consider the system of ODE

$$\frac{d}{dx}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -0.5y \\ 4-0.3z-0.1y \end{pmatrix}, \begin{pmatrix} y(0)=4 \\ z(0)=6 \end{pmatrix}$$

The theoretical solution is given by

$$y(x) = 4e^{-0.5x}$$

$$z(x) = \frac{-4e^{-x/2}+40}{3} - 6$$

TABLE 3. Maximum of Absolute Errors for Problem 4.3

x	*Okunuga & Ohigie [17]*	*4th Order ETRs*	*4th Order ETR_2s*
0.1	3.71e-008	2.978592172908634e-008	2.978592172908634e-008
0.2	2.34 e-008	5.666648972280086e-008	5.666648972280086e-008
0.3	9.19 e-007	8.085424862969148e-008	8.085424818560227e-008
0.4	8.87 e-007	1.025479199334711e-007	1.025479190452927e-007
0.5	7.49 e-007	1.219332483870517e-007	1.219332470547840e-007
0.6	6.04 e-007	1.391837916031591e-007	1.391837902708915e-007
0.7	4.93 e-007	1.544616710091873e-007	1.544616696769197e-007
0.8	4.12 e-007	1.679182695113468e-007	1.679182686231684e-007
0.9	3.73 e-007	1.796948976284796e-007	1.796948971843904e-007
1.0	2.19 e-007	1.899234152169527e-007	1.899234147728635e-007

Problem 4**:** To test the efficiency of the proposed algorithm we used the following stiff initial value problem arising from the biochemistry see [16].

$$\frac{dy_1(t)}{dt} = \frac{1}{\alpha}(y_1(t) + y_2(t) - y_1(t)y_2(t) - qy_1^2(t)),$$

$$\frac{dy_2(t)}{dt} = 2my_3(t) - y_2(t) - y_1(t)y_2(t),$$

$$\frac{dy_3(t)}{dt} = \frac{1}{r}(y_1(t) - y_3(t)),\ y_1(0) = a, y_2(0) = b, y_3(0) = d$$

Here α, m, q and r are some parameters, a, b and d are the initial values. For some values of parameters this model has a periodic solution very sensitive for the parameter values. Let the parameter values be as follows: $\alpha := 0.1;\ q := 0.01;\ m := 0.5;\ r := 1$ and the initial conditions are $a = 0,\ b = 0.5$ and $d = 0.8$. The test problem was solved on the interval [0, 30]. Problem 4 was extracted from the work of [15].

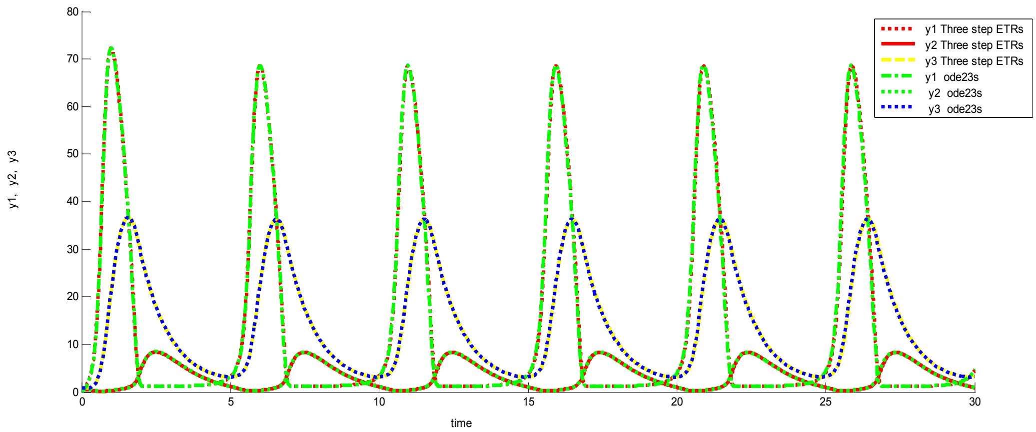

Figure 2. Solution curve for Problem 4.4 using *4th Order* ETRs

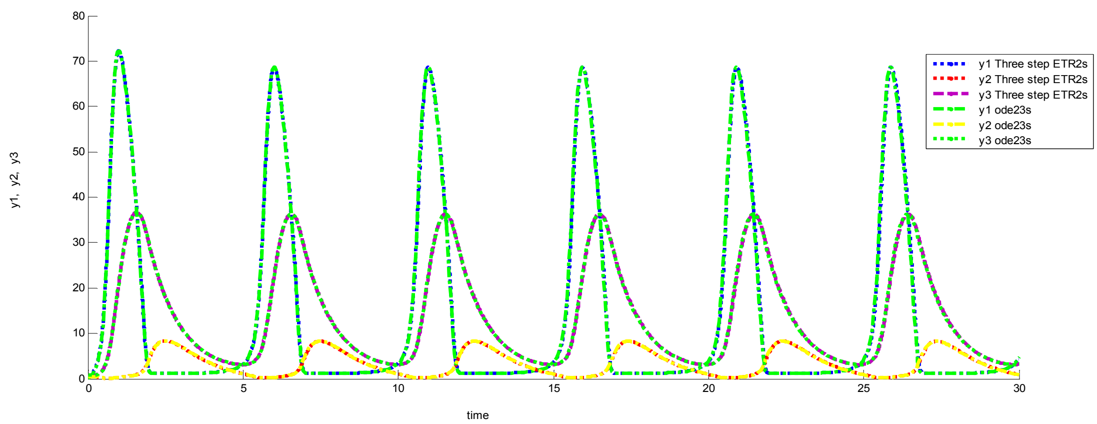

Figure 3. Solution curve for Problem 4.4 using *4th Order* ETR$_2$s

Problem 5**:**

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 & -2\times10^{-3} y_1 \\ 6\times10^{-4} y_2 & -0.5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_1(0) = 10^3 \\ y_2(0) = 2\times10^2 \end{pmatrix}$$

This fifth example has been reported by [20] and is a real life problem of mathematical models for predicting the population dynamics of competing species [3].

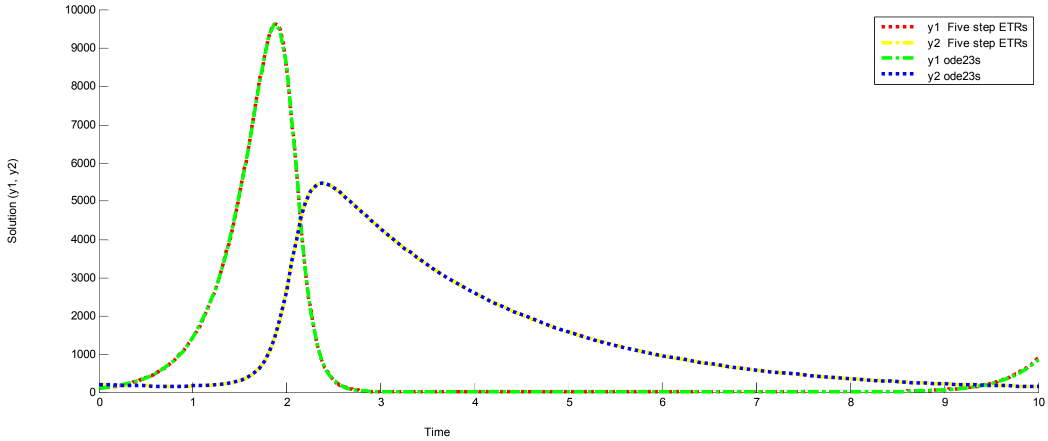

Figure 4. Solution curve for Problem 4.5 using *6th Order* ETRs

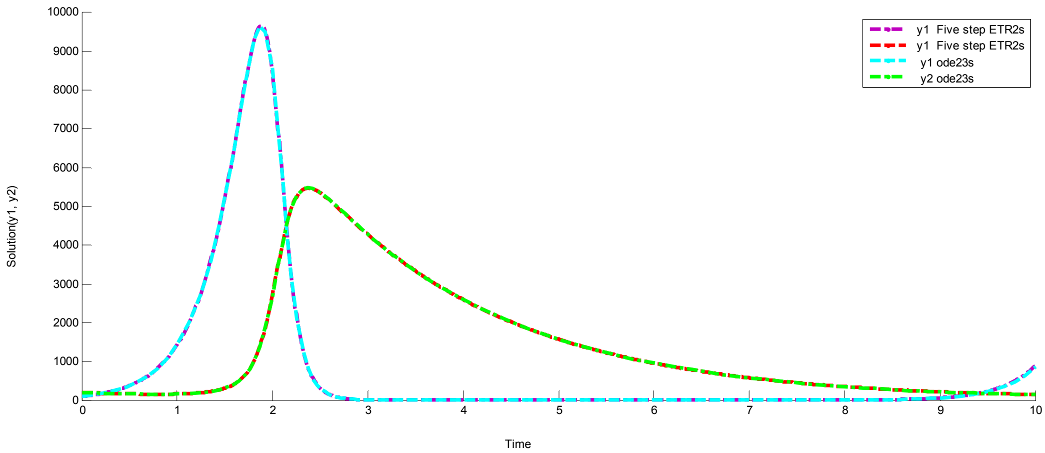

Figure 5. Solution curve for Problem 4.5 using *6th Order* ETR_2s

Table 4. Coefficients of ETRS/ETR$_2$S

ETRs					
v	p	η_k	β_0	β_1	β_2
2	4	24	-1	13	
3	6	1440	11	-93	802
ETR$_2$s					
			α_0	α_1	α_2
2	4	12	-1	-9	
3	6	120	1	15	-80

Table 5. Error Constants of ETRS/ ETR$_2$S

ETRs	
Eqn.	C_{p+1}
15	$\left(\frac{11}{720}, -\frac{19}{720}, -\frac{1}{90}\right)^T$
21	$\left(-\frac{37}{3780}, \frac{271}{60480}, -\frac{191}{60480}, \frac{1}{756}, -\frac{29}{2240}\right)^T$
ETR$_2$s	
20	$\left(\frac{1}{120}, -\frac{1}{6}, -\frac{1}{60}\right)^T$
22	$\left(-\frac{1}{840}, -\frac{187}{840}, -\frac{1}{240}, \frac{1}{840}, \frac{1}{70}\right)^T$

V. CONCLUSION

We have derived a class of fourth and sixth order Extended Trapezoidal Rule of First (ETRs) and Second kind (ETR$_2$s). Our newly derived methods in block form are shown to have extensive regions of stability and in particular are A-stable up to order 6 and so very suitable for stiff system of ordinary differential equations. The continuous formulation for each step number k is evaluated at the end point of the interval to recover the discrete schemes of [4] as special case. To this end, the idea of additional conditions is discarded. Furthermore, we do not need any pair in our implementation. Our block methods preserve the A-stability property of the trapezoidal rule (refer to figure 1), they are also less expensive in terms of the number of functions evaluation per step. Consequently, our methods were shown to compete favorably with the well known MATLAB Ode solver (Ode 23s). We evaluated their performance on a set of five challenging systems of first order stiff ordinary differential equations and compared their performance with some existing theoretical solutions. The numerical results are quite satisfactory given a better accuracy.

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