# Groupoids of Left Quotients

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***Abstract-*** **A subcategory $\mathbf{C}$ of a groupoid $\mathbb{G}$ is a left order in $\mathbb{G}$, if every element of $\mathbb{G}$ can be written as $a^{-1}b$ where $a, b \in \mathbf{C}$. We give a characterization of left orders in groupoids.**

***Keywords*: groupoid of fractions, I-order, *I*-quotients, right cancellative monoid.**

## I. Introduction

In this article we investigate left orders in groupoids. This work is part of a continuing investigation of categories of quotients. The motivation for our investigation comes from semigroups of quotients and categories of fractions. Our purpose is the investigation of a similar problem in groupoid theory.

Fountain and Petrich introduced the notion of a completely 0-simple semigroup of quotients in [3]. It is well-known that groupoids are generalisations of groups, also, inverse semigroups can be regarded as special kinds of ordered groupoids. The concept of semigroups of quotients extends that of a group of quotients, introduced by Ore-Dubreil. We recall that a group $G$ is a *group of left quotients* of its subsemigroup $S$ if every element of $G$ can be written as $a^{-1}b$ for some $a, b \in S$.

The author and Gould [6] have extended the classical notion of left orders in inverse semigroups. They have introduced the following definition: Let $Q$ be an inverse semigroup. A subsemigroup $S$ of $Q$ is a *left I-order* in $Q$ and $Q$ is a *semigroup of left I-quotient* of $S$, if every element of $Q$ can be written as $a^{-1}b$ where $a, b \in S$ and $a^{-1}$ is the inverse of a in the sense of inverse semigroup theory. The notions of *right I-order* and semigroup *of right I-quotients* are defined dually. If $S$ is both a left and a right I-order in an inverse semigroup $Q$, we say that $S$ is an *I-order* in $Q$ and $Q$ is a semigroup of *I-quotients* of $S$. If we insist on $a$ and $b$ being $\mathcal{R}$-related in $Q$, then we say that $S$ is a *straight left I-order in $Q$*.

The theory of categories of fractions was developed by Gabriel and Zisman [5]. The key idea is that starting with a category $\mathbf{C}$ we can associate a groupoid to $\mathbf{C}$ by adding all the inverses of all the elements of $\mathbf{C}$ to $\mathbf{C}$. We then produce a groupoid $\mathbb{G}(\mathbf{C}) = \mathbf{C}^{-1}\mathbf{C}$ and a functor $\boldsymbol{\iota}: \mathbf{C} \to \mathbb{G}$ such that $\mathbb{G}(\mathbf{C})$ is generated by $\boldsymbol{\iota}(\mathbf{C})$, we call $\mathbb{G}$ a *category of fractions*. Tobias in [4] showed that for any category with conditions which are analogues of the Ore condition in the theory of non-commutative rings (see, [10]), there is a groupoid of fractions.

Now, we are in a position to define a groupoid of left quotients. Let $\mathbf{C}$ be a subcategory of a groupoid $\mathbb{G}$. We say that $\mathbf{C}$ is a *left order* in $\mathbb{G}$ or $\mathbb{G}$ is a groupoid of *left quotients* of $\mathbf{C}$ if every element of $\mathbb{G}$ can be written as $a^{-1}b$ for some $a, b \in \mathbf{C}$. *Right orders* and *groupoids of right quotients* are defined dually. If $\mathbf{C}$ is both a left and a right order in $\mathbb{G}$, then $\mathbf{C}$ is an *order* in $\mathbb{G}$ and $\mathbb{G}$ is a *groupoid of quotients* of $\mathbf{C}$.

This work is divided up into three sections. In Section 1 we summarize the background on groupoids and inverse semigroups that we shall need throughout the article. A Theorem 1.24 in [2] due to Ore and Dubreil shows that a semigroup S has a group of left quotients if and only if it is *right reversible*, that is, $Sa \cap Sb \neq \emptyset$ for all $a, b \in S$ and $S$ is cancellative. In Section 2 we prove the category version of such a theorem. We stress that this work is not new - it has been studied by a number of authors, by using the notion of category as a collection of objects and arrows. We regard a small category as a generealisation of a monoid to prove such a theorem. Consequently, the relationship between the groupoids of left quotients and inverse semigroups of left I-quotients becomes clearer. In Section 3 we show that a groupoid of left quotients is unique up to isomorphism.

## II. Preliminaries and notions

In this section we set up the definitions and results about groupoids and inverse semigroups. Standard references include [2] for inverse semigroups, and [7] for groupoids.

There are two definitions of (small) categories. The first one in [7] considers the category as a collection of objects (sets) and homomorphisms between them satisfying certain conditions. A *category*, consists of a set of objects $\{a, b, c, \dots\}$ and homomorphisms between the objects such that:

($i$) Homomorphisms are composable: given homomorphisms $a: u \to v$ and $b: v \to w$, the homomorphism $ab: u \to w$ exists, otherwise $ab$ is not defined;

($ii$) Composition is associative: given homomorphisms $a: u \to v, b: v \to w$ and $c: w \to z$, $(ab)c = a(bc)$;

($iii$) Existence of an identity homomorphism: For each object $u$, there is an identity homomorphism $e_u: u \to u$ such that for any homomorphism $a: u \to v,\ e_u a = a = a e_v$.

A category $\mathbf{T}$ is called a *subcategory* of the category $\mathbf{C}$, if the objects of $\mathbf{T}$ are also objects of $\mathbf{C}$, and the homomorphisms of $\mathbf{T}$ are also homorphisms of $\mathbf{C}$ such that

($i$) for every $u$ in $\mathrm{Ob}(\mathbf{T})$, the identity homomorphism $e_u$ is in Hom $\mathbf{T}$;

($ii$) for every pair of homomorphisms $f$ and $g$ in Hom $\mathbf{T}$ the composite $fg$ is in Hom $\mathbf{T}$ whenever it is defined.

The second definition regards the category as an algebraic structure in its own right. In this definition we can look at categories as generalisations of monoids. Let $\mathbf{C}$ be a set equipped with a partial binary operation which we shall denote by $\cdot$ or by concatenation. If $x, y \in \mathbf{C}$ and the product $x \cdot y$ is defined we write $\exists x \cdot y$. An element $e \in \mathbf{C}$ is called an *identity* if $\exists e \cdot x$ implies $e \cdot x = x$ and $\exists x \cdot e$ implies $x \cdot e = x$. The set of identities of $\mathbf{C}$ is denoted by $\mathbf{C_0}$. The pair $(\mathbf{C}, \cdot)$ is said to be a *category* if the following axioms hold:

(C1): $x \cdot (y \cdot z)$ exists if, and only if, $(x \cdot y) \cdot z$ exists, in which case they are equal.

(C2): $x \cdot (y \cdot z)$ exists if, and only if, $x \cdot y$ and $y \cdot z$ exist.

(C3): For each $x \in \mathbf{C}$ there exist identities $e$ and $f$ such that $\exists x \cdot e$ and $\exists f \cdot x$.

It is convenient to write $xy$ instead of $x \cdot y$. From axiom (C3), it follows that the identities $e$ and $f$ are uniquely determined by $x$. We write $e = \mathbf{r}(x)$ and $f = \mathbf{d}(x)$. We call $\mathbf{d}(x)$ the *domain* of $x$ and $\mathbf{r}(x)$ the *range* of $x$. Observe that $\exists xy$ if, and only if $\mathbf{r}(x) = \mathbf{d}(y)$; in which case $\mathbf{d}(xy) = \mathbf{d}(x)$ and $\mathbf{r}(xy) = \mathbf{r}(y)$. The elements of $\mathbf{C}$ are called *homomorphisms*.

A *subcategory* $\mathbf{T}$ of a category $\mathbf{C}$ is a collection of some of the identities and some of the homomorphisms of $\mathbf{C}$ which include with each homomorphism, $a$, both $\mathbf{d}(a)$ and $\mathbf{r}(a)$, and with each composable pair of homomorphisms in $\mathbf{T}$, their composite. In other words, $\mathbf{T}$ is a category in its own right.

The two definitions are equivalent. The first one can be easily turned into the second one and vice versa. A homomorphism a is said to be an isomorphism if there exists an element $a^{-1}$ such that $\mathbf{r}(a) = a^{-1}a$ and $\mathbf{d}(a) = aa^{-1}$.

A *groupoid* $\mathbb{G}$ is a category in which every element is an isomomorhism. A group may be thought of as a one-object groupoid. A category $\mathbb{G}$ is said to be *connected* if for each $e, f \in \mathbb{G}_0$ there is an element $x$ with $\mathbf{d}(x) = e$ and $\mathbf{r}(x) = f$. Connected groupoids are known as *Brandt groupoids*.

If $\mathbb{G}$ and $\mathbb{P}$ are categories, then $\varphi: \mathbb{G} \to \mathbb{P}$ is a *homomorphism* if $\exists xy$ implies that $(xy)\varphi = (x\varphi)(y\varphi)$ and for all $x \in \mathbb{G}$ we have that $(\mathbf{d}(x))\varphi = \mathbf{d}(x\varphi)$ and $(\mathbf{r}(x))\varphi = \mathbf{r}(x\varphi)$. In case where $\mathbb{G}$ and $\mathbb{P}$ are groupoids we have that $\varphi: \mathbb{G} \to \mathbb{P}$ is a homomorphism if $\exists xy$ implies that $(xy)\varphi = (x\varphi)(y\varphi)$ and so $x^{-1}\varphi = (x\varphi)^{-1}$.

The following lemma gives useful properties of groupoids which will be used without further mention. Proofs can be found in [8].

**Lemma 1.1.** Let $\mathbb{G}$ be a groupoid. Then for any $x, y \in \mathbb{G}$ we have

($i$) For all $x \in \mathbb{G}$ we have $\mathbf{r}(x^{-1}) = \mathbf{d}(x)$ and $\mathbf{d}(x^{-1}) = \mathbf{r}(x)$.

($ii$) If $\exists xy$, then $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$and $(xy)^{-1} = y^{-1}x^{-1}$.

($iii$) $(x^{-1})^{-1} = x$ for any $x \in \mathbb{G}$.

From now on we shall adopt the second definition of categories. In other words, we regard categories as a generalisation of monoids.

**Proposition 1.2.** [12] Let $G$ be a group and $I$ a non-empty set. Define a partial product on $\mathrm{I} \times G \times I$ by $(i, g, j)(j, h, k) = (i, gh, k)$ and undefined in all other cases. Then $\mathrm{I} \times G \times I$ is a connected groupoid, and every connected groupoid is isomorphic to one constructed in this way.

A *Brandt semigroup* is a completely 0-simple inverse semigroup. By Theorem II.3.5 in [12] every Brandt semigroup is isomorphic to $\mathrm{B(G, I)}$ for some group $G$ and non-empty set I where $\mathrm{B(G, I)}$ is constructed as follows:

As a set $\mathrm{B(G, I)} = (\mathrm{I} \times G \times I) \cup \{0\}$ the binary operation is defined by

$$(i, a, j)(k, b, l) = \begin{cases} (i, ab, l), & if\ j = k; \\ 0, & else \end{cases}$$

and

$$(i, a, j)0 = 0(i, a, j) = 00 = 0.$$

In [2] it is shown that if we adjoin 0 to a Brandt groupoid B, defining $xy = 0$ if $xy$ is undefined in B, we get a Brandt semigroup $\mathrm{B}^0$.

Let $\{S_i : i \in I\}$ be a family of disjoint semigroups with zero, and put $S_i{}^* = S \setminus \{0\}$. Let $S = \bigcup_{i \in I} S_i^* \cup 0$ with the multiplication

$$a * b = \begin{cases} ab\ \ if\ a, b \in S_i\ for\ some\ i\ and\ ab \neq 0\ in\ S_i; \\ 0, \qquad else. \end{cases}$$

With this multiplication S is a semigroup called a *0-direct union* of the $S_i$.

An inverse semigroup $S$ with zero is a *primitive inverse* semigroup if all its nonzero idempotents are primitive, where an idempotent $e$ of $S$ is called *primitive* if $e \neq 0$ and $f \leq e$ implies $f = 0$ or $e = f$. Note that every Brandt semigroup is a primitive inverse semigroup.

**Theorem 1.3.** [12] Brandt semigroups are precisely the connected groupoids with a zero adjoined, and every primitive inverse semigroup with zero is a 0-direct union of Brandt semigroups.

Notice that a groupoid is a disjoint union of its connected components.

**Theorem 1.4.** [12] Let $\mathbb{G}$ be a groupoid. Suppose that $0 \notin \mathbb{G}$ and put $\mathbb{G}^0 = \mathbb{G} \cup \{0\}$. Define a binary operation on $\mathbb{G}^0$ as follows: if $x, y \in \mathbb{G}$ and $\exists x \cdot y$ in the groupoid $\mathbb{G}$, then $xy = x \cdot y$; all other products in $\mathbb{G}^0$ are 0. With this operation $\mathbb{G}^0$ is a primitive inverse semigroup.

**Theorem 1.5.** [12] Let $S$ be an inverse semigroup with zero. Then $S$ is primitive if, and only if, it is isomorphic to a groupoid with zero adjoined.

In [2], it is shown that every primitive inverse semigroup with zero is a 0-direct union of Brandt semigroups. An *ordered groupoid* $(\mathbb{G}, \leq)$ is a groupoid $\mathbb{G}$ equipped with a partial order $\leq$ satisfies the following axioms:

**(OG1)** If $x \leq y$ then $x^{-1} \leq y^{-1}$.

**(OG2)** If $x \leq y$ and $\acute{x} \leq \acute{y}$ and the products $x\acute{x}$ and $y\acute{y}$ are defined then $x\acute{x} \leq y\acute{y}$.

**(OG3)** If $e \in \mathbb{G}_0$ is such that $e \leq \mathbf{d}(x)$ there exists a unique element $(e|x) \in \mathbb{G}$, called the *restriction* of $x$ to $e$, such that $(e|x) \leq x$ and $\mathbf{d}(e|x) = e$.

**(OG3)*** If $e \in \mathbb{G}_0$ is such that $e \leq \mathbf{r}(x)$ there exists a unique element $(x|e) \in \mathbb{G}$, called the *corestriction* of $x$ to $e$, such that $(x|e) \leq x$ and $\mathbf{r}(x|e) = e$.

In fact, it is shown in [12] that axiom (**OG3**)* is a consequence of the other axioms.

A partially ordered set $X$ is called a *meet semilattice* if, for every $x, y \in X$, there is a greatest lower bound $x \wedge y$. An ordered groupoid is *inductive* if the partially ordered set of identities forms a meet-semilattice. An ordered groupoid $\mathbb{G}$ is said to be $*$-*inductive* if each pair of identities that has a lower bound has a greatest lower bound. We can look at any inverse semigroup as an inductive groupoid; the order is the natural order and the multiplication is the usual multiplication.

We shall now describe the relationship between inverse semigroups and inductive groupoids. We begin with the following definition.

**Definition 1.1**. For an arbitrary inverse semigroup $S$, the *restricted product* (also called the `trace product') of elements $x$ and $y$ of $S$ is $xy$ if $x^{-1}x = yy^{-1}$ and undefined otherwise.

Let $S$ be an inverse semigroup with the natural partial order $\leq$. Define a partial operation $\circ$ on $S$ as follows:

$$x \circ y \text{ defined iff } x^{-1}x = yy^{-1}$$

in which case $x \circ y = xy$. Then $\mathbb{G}(S) = (S, \circ)$ is a groupoid and $\mathcal{G}(S) = (S, \circ, \leq)$ is an inductive groupoid with $(x|e) = xe$ and $(e|x) = ex$ and $e = x^{-1}xyy^{-1}$.

**Definition 1.2.** Let $(\mathbb{G}, \cdot, \leq)$ be an ordered groupoid and let $x, y \in \mathbb{G}$ are such that $e = \mathbf{r}(x) \wedge \mathbf{d}(y)$ is defined. Then the *pseudoproduct* of $x$ and $y$ is defined as follows:

$$x \otimes y = (x|e)(e|y).$$

If $(\mathbb{G},\cdot,\leq)$is an inductive groupoid, then $\boldsymbol{S}(\mathbb{G}) = (\mathbb{G},\otimes)$ is an inverse semigroup associated of $\mathbb{G}$ having the same partial order as $\mathbb{G}$ such that the inverse of any element in $(\mathbb{G},\cdot,\leq)$ coincides with the inverse of the same element in $\boldsymbol{S}(\mathbb{G})$. The pseudoproduct is everywhere defined in $\boldsymbol{S}(\mathbb{G})$ and coincides with the product $\cdot$ in $\mathbb{G}$ whenever $\cdot$ is defined, that is, if $\exists x.y$, then $x \otimes y = x \cdot y$.

It is noted in [12] that in an inductive groupoid $\mathbb{G}$, for $a \in \mathbb{G}$ and $e \in \mathbb{G}_0$ with $e \leq \mathbf{r}(a)$, the corestriction $e|a$ is given by $(e|a) = (a^{-1}|e)^{-1}$. By Linking this with the inverse semigroup which associated to $\mathbb{G}$, we present a short proof in the following lemma.

**Lemma 1.6.** Let $(\mathbb{G},\cdot,\leq)$ be an inductive groupoid associated to an inverse semigroup $(\mathbb{G},\otimes)$. If $a \in \mathbb{G}$ and $e \in \mathbb{G}_0$ with $e \leq \mathbf{d}(a)$, then $(a^{-1}|e)^{-1} = (e|a)$.

**Proof.** First we show that $(a^{-1}|e)$ exists. As $e \leq \mathbf{d}(a)$ and $\mathbf{d}(a) = \mathbf{r}(a^{-1})$ we have that $e \leq \mathbf{r}(a^{-1})$ . Hence by (**OG3**) $(a^{-1}|e)$ exists. To show that $(a^{-1}|e)$ is the inverse of $(e|a)$. We note that

$$(a^{-1}|e)(e|a)(a^{-1}|e) = (a^{-1}e)(ea)(a^{-1}e) = a^{-1}e = (a^{-1}|e).$$

Also,

$$(e|a)(a^{-1}|e)(e|a) = (ea)(a^{-1}e)(ea) = ea = (e|a).$$

We recall that a semigroup $Q$ with zero is defined to be *categorical at 0* if whenever $a, b, c \in Q$ are such that $ab \neq 0$ and $bc \neq 0$, then $abc \neq 0$. The set of non-zero elements of a semigroup $S$ will be denoted by $S^*$.

Let $Q$ be an inverse semigroup which is categorical at zero. Define a partial binary operation $\circ$ on $Q^*$ by

$$a \circ b = \begin{cases} ab, & if\ \ a^{-1}a = bb^{-1}; \\ undefined, & otherwise. \end{cases}$$

It is easy to see that (C1) holds. Assume that $a \circ b$ and $b \circ c$ are defined in $Q^*$ so that $ab \neq 0$ and $bc \neq 0$ in $Q$. As $Q$ categorical at 0 we have $abc \neq 0$ so that $a \circ (b \circ c)$ is defined in $Q^*$. On the other hand, if $a \circ (b \circ c)$ exists in $Q^*$, then $b^{-1}b = cc^{-1}$and $a^{-1}a = (bc)(bc)^{-1} = bcc^{-1}b^{-1} = bb^{-1}$. Hence $a \circ b$ and $b \circ c$ exist. Thus (C2) holds. For any $a \in Q^*$ the identities $\mathbf{d}(a) = aa^{-1}$ and $\mathbf{r}(a) = a^{-1}a$ satisfy (C3). Hence $Q^*$ is a category and any element a in $Q^*$ has the same inverse $a^{-1}$ as in $Q$. We have

**Lemma 1.7.** Let $\boldsymbol{Q}$ be an inverse semigroup with zero. If $Q$ categorical at 0, then $Q^* = Q \setminus \{0\}$ is a groupoid.

Following [11], we define Green's relations on an ordered groupoid $(\mathbb{G},\cdot,\leq)$. First we define useful subsets of $\mathbb{G}$. For a subset H of $\mathbb{G}$, we define $(H]$ as follows:

$$(H] = \{t \in \mathbb{G} : \mathrm{t} \leq h \text{ for some } h \in H\}.$$

For $a, b \in \mathbb{G}$, put

$$\mathbb{G}a = \{xa : x \in \mathbb{G} \text{ and } \exists xa\}.$$

We define $a\mathbb{G}$ and $a\mathbb{G}b$ similarly.

A nonempty subset $I$ of $\mathbb{G}$ is called a right (left ) ideal of $\mathbb{G}$ if

(1) $I\mathbb{G} \subseteq I$ ($\mathbb{G}I \subseteq I$)

(2) if $a \in I$ and $b \leq a$, then $b \in I$.

We say that $I$ is an *ideal* of $\mathbb{G}$ if it is both a right and a left ideal of $\mathbb{G}$. We denote by $R(a), L(a), I(a)$ the right ideal, left ideal, ideal of $\mathbb{G}$, respectively, generated by $a$ ($a \in \mathbb{G}$). For each $a \in \mathbb{G}$, we have

$$R(a) = (a \cup a\mathbb{G}], L(a) = (a \cup \mathbb{G}a] \text{ and } I(a) = (a \cup a\mathbb{G} \cup \mathbb{G}a \cup \mathbb{G}a\mathbb{G}].$$

For an ordered groupoid $\mathbb{G}$, the Green's relations $\mathcal{R}, \mathcal{L}$ and $\mathcal{J}$ defined on $\mathbb{G}$ by

$$a\,\mathcal{R}\,b \iff R(a) = R(b);$$
$$a\,\mathcal{L}\,b \iff L(a) = L(b);$$
$$a\,\mathcal{J}\,b \iff J(a) = J(b).$$

It is straightforward to show that a $a\mathbb{G} = \mathbf{d}(a)\mathbb{G}$ ($\mathbb{G}a = \mathbb{G}\mathbf{r}(a)$) for all $a$ in $\mathbb{G}$.

**Lemma 1.8.** Let $\mathbb{G}$ be an ordered groupoid and let $a, b \in \mathbb{G}$. Then

(1) $a\,\mathcal{R}\,b \iff \boldsymbol{d}(a) = \boldsymbol{d}(b)$.

(2) $a\,\mathcal{L}\,b \iff \boldsymbol{r}(a) = \boldsymbol{r}(b)$.

**Proof.** Suppose that $R(a) = R(b)$ it is clear that if $a = b$ we have that $a\,\mathcal{R}\,b$. If $a \neq b$ then $a \in R(b)$ so that $a \in (b \cup b\mathbb{G}] = \{t \in \mathbb{G}: t \leq h$ for some $h \in b \cup b\mathbb{G}\}$. It is easy to see that $aa^{-1} \leq bb^{-1}$. Hence $\mathbf{d}(b) \leq \mathbf{d}(a)$. Similarly, we can show that $\mathbf{d}(b) \leq \mathbf{d}(a)$. Thus $\mathbf{d}(a) = \mathbf{d}(b)$.

Conversely, suppose that $\mathbf{d}(a) = \mathbf{d}(b)$. Let $x \in R(a) = (a \cup a\mathbb{G}]$ so that $x \leq h$ for some $h \in a \cup a\mathbb{G}$ so that $h = a$ or $h \in a\mathbb{G} = \mathbf{d}(a)\mathbb{G} = \mathbf{d}(b)\mathbb{G} = b\mathbb{G}$. In the latter case, it is clear that $x \in R(b)$. In the former case, $\mathbf{d}(h) = \mathbf{d}(a)$ and as $x \leq h$ we have that $xx^{-1} \leq hh^{-1} = \mathbf{d}(h) = \mathbf{d}(a) = \mathbf{d}(b)$ and so $x \leq \mathbf{d}(a)x$. Since $\mathbf{d}(a)x \in b\mathbb{G} \subseteq b \cup b\mathbb{G}$ we have that $x \in R(b)$ and so $R(a) \subseteq R(b)$. Similarly, $R(b) \subseteq R(a)$. Thus $R(b) = R(a)$ as required.

## III. Left orders in groupoids

In this section we consider the relationship between left orders in an inductive groupoid $\boldsymbol{\mathcal{G}}$ and left I-orders in $\boldsymbol{\mathcal{S}}(\boldsymbol{\mathcal{G}})$. We give a characterization of left orders in groupoids. By using the second definition of categories we prove the category version of theorem due to Ore-Dubreil mentioned in the introduction.

A category is said to be *right* (*left*) *cancellative* if $\exists x \cdot a, \exists y \cdot a$ ($\exists a \cdot x, \exists a \cdot y$) and $xa = ya$ implies $x = y$ ($ax = ay$ implies $x = y$). A *cancellative category* is one which is both left and right cancellative.

Following [9], a category $\mathbf{C}$ is said to be *right reversible* if for all $a, b \in \mathbf{C}$, with $\mathbf{r}(a) = \mathbf{r}(b)$, there exist $x, y \in \mathbf{C}$ such that $xa = yb$. In diagrammatic this just

$$\begin{tikzcd} \cdot \arrow[r, "x"] \arrow[d, "y"'] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "a"'] & \cdot \end{tikzcd}$$

Figure 2.1

Let $\mathbf{C}$ be a category and $a, b \in \mathbf{C}$ such that $\mathbf{d}(a) = \mathbf{d}(b)$ we say that $a$ and $b$ have a *pushout*, if $ax = by$ for some $x, y \in \mathbf{C}$.

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "b"'] & \cdot \arrow[d, "x"] \\ \cdot \arrow[r, "y"'] & \cdot \end{tikzcd}$$

Figure 2.2

**Remark 2.1.** If a category $\mathbf{C}$ is a left order in a groupoid $\mathbb{G}$, then any element in $\mathbb{G}$ has the form $a^{-1}b$. It is clear that $\mathbf{d}(a) = \mathbf{d}(b)$ and $a^{-1}\, \mathcal{R}\, a^{-1}b\, \mathcal{L}\, b$. If any two elements in $\mathbf{C}$ have a pushout, then $\mathbf{C}$ is a right order in $\mathbb{G}$. Hence $\mathbf{C}$ is an order in $\mathbb{G}$. We have the following diagram

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "b"'] & \cdot \arrow[d, "v"] \arrow[dl, "q"'] \\ \cdot \arrow[r, "u"'] & \cdot \end{tikzcd}$$

Figure 2.3

**Lemma 2.1.** A category $\mathbf{C}$ is a left order in an inductive groupoid $\mathcal{G}$ if and only if ($\mathbf{C}$,$\otimes$) is a left I-order in ($\mathcal{G}$,$\otimes$) such that $a \otimes a^{-1}, a^{-1} \otimes a \in \mathbf{C}$ for all $a \in \mathbf{C}$.

**Proof.** Suppose that $\mathbf{C}$ is a left order in $\mathcal{G}$. For any $q \in \mathcal{G}$, there are $a, b \in \mathbf{C}$ such that for $e = aa^{-1}bb^{-1}$ we have

$$\begin{aligned} q &= a^{-1}b \\ &= a^{-1}aa^{-1}bb^{-1}b \\ &= (a^{-1}e)(eb) \\ &= (ea)^{-1}(eb) \\ &= (e|a)^{-1}(e|b) \\ &= (a^{-1}|e)(e|b) \\ &= a^{-1} \otimes b. \end{aligned}$$

We aim to show that the pseudoproduct is everywhere defined in $\mathbf{C}$ and coincides with the product $\cdot$ in $\mathbf{C}$ whenever $\cdot$ is defined. If $a, b \in \mathbf{C}$ and $\exists\, a.b$, then $\mathbf{r}(a) = \mathbf{d}(b)$. Hence

$$\begin{aligned} a \otimes b &= (a|e)(e|b) \qquad \text{where } e = \mathbf{r}(a) \wedge \mathbf{d}(b) = \mathbf{r}(a) = \mathbf{d}(b) \\ &= (a|\mathbf{r}(a))(e|\mathbf{d}(b)) \\ &= (a\mathbf{r}(a)(\mathbf{d}(b)b) \\ &= a.b. \end{aligned}$$

As $\mathbf{C}$ is a subcategory of $\mathcal{G}$ we have $a \otimes b = a.b \in \mathbf{C}$. Hence ($\mathbf{C}$,$\otimes$) is a subsemigroup of ($\mathcal{G}$,$\otimes$).

Let $a$ be any element of $\mathbf{C}$, and let $e = aa^{-1}$. Then

$$\mathbf{r}(a) = a^{-1}a = a^{-1}aa^{-1}a = a^{-1}ea = (a^{-1}e)(ea) = (a^{-1}|e)(e|a) = a^{-1} \otimes a.$$

Since $\mathbf{C}$ is a subcategory of $\mathcal{G}$ we have $a^{-1} \otimes a = a^{-1}a = \mathbf{r}(a) \in \mathbf{C}$. Similarly, $a \otimes a^{-1} = aa^{-1} = \mathbf{d}(a) \in \mathbf{C}$. The converse follows by reversing the argument.

**Lemma 2.2.** Let $S$ be a semigroup which is a straight left I-order in an inverse semigroup $Q$. On the set $Q$ define a partial product $\circ$. Then ($S \cup E(Q)$,$\circ$) is a left order in ($Q$,$\circ$).

**Proof.** Suppose that $S$ is a straight left I-order in Q. For any $q \in Q$, there are $c, d \in S$ such that $q = c^{-1}d$ with $c\, \mathcal{R}\, d$ so that $cc^{-1} = dd^{-1}$. Hence $q = c^{-1}d$ is defined in ($Q$,$\circ$) and $c, d \in S \cup E(Q)$. It is easy to see that $E(Q) = \{aa^{-1} : a \in S\}$. Let $a^{-1}a \in E(Q)$ for some $a \in S$ and let $b \in S$ such that $ba^{-1}a$ is defined in ($Q$,$\circ$) so that $b^{-1}b = a^{-1}a$. Hence $b = bb^{-1}b = ba^{-1}a \in S$. Similarly, if $a^{-1}ab$ is defined, then $a^{-1}a = bb^{-1}$ and so $b = bb^{-1}b = a^{-1}ab \in S$. Thus ($S \cup E(Q)$,$\circ$) is a left order in ($Q$,$\circ$).

The following lemmas give a characterisation for categories which are left orders in groupoids. The proofs of such lemmas are quite straightforward and it can be deduced from [5] and [4], but we give it for completeness.

**Lemma 2.3.** Let $\mathbf{C}$ be a left order in a groupoid $\mathbb{G}$. Then

($i$) $\mathbf{C}$ is cancellative;

($ii$) $\mathbf{C}$ is right reversible;

($iii$) any element in $\mathbb{G}_0$ has the form $a^{-1}a$ for some $a \in \mathbf{C}$. Consequently, $\mathbf{C_0} = \mathbb{G}_0$.

**Proof.** ($i$) This is clear.

($ii$) Let $a, b \in \mathbf{C}$ with $\mathbf{r}(a) = \mathbf{r}(b)$ so that $ab^{-1}$ is defined in $\mathbb{G}$. Since $\mathbb{G}$ is a category of left quotients of $\mathbf{C}$, we have that $ab^{-1} = x^{-1}y$ where $x, y \in \mathbf{C}$ and $\mathbf{d}(x) = \mathbf{d}(y)$. Then

$$xa = xab^{-1}b = xx^{-1}yb = yb.$$

($iii$) Let $e$ be an identity in $\mathbb{G}_0$. As $\mathbf{C}$ is a left order in $\mathbb{G}$ we have that $e = a^{-1}b$ for some $a, b \in \mathbf{C}$ so that $\mathbf{d}(a) = \mathbf{d}(b)$. Since $e$ is identity and $\mathbf{d}(a) = \mathbf{d}(b)$ we have

$$a = ae = aa^{-1}b = \mathbf{d}(b)b = b.$$

Hence $e = a^{-1}a = \mathbf{r}(a) \in \mathbf{C_0}$ so that $\mathbb{G}_0 \subseteq \mathbf{C_0}$. Thus $\mathbf{C_0} = \mathbb{G}_0$.

**Lemma 2.4.** Suppose that $\mathbb{G}$ is a groupoid of left quotients of $\mathbf{C}$. Then for all $a, b, \mathrm{c}, \mathrm{d} \in \mathbf{C}$ the following are equivalent:

$(i)$ $a^{-1}b = c^{-1}d$;

($ii$) there exist $x, y \in \mathbf{C}$ such that $xa = yc$ and $xb = yd$;

($iii$) $\mathbf{r}(a) = \mathbf{r}(c)$, $\mathbf{r}(b) = \mathbf{r}(d)$ and for all $x, y \in \mathbf{C}$ we have $xa = yc \iff xb = yd$.

**Proof.**$(i) \Rightarrow (ii)$. Suppose that $a^{-1}b = c^{-1}d$ for $a, b, c, d \in \mathbf{C}$ so that $\mathbf{r}(a) = \mathbf{r}(c)$ and $\mathbf{r}(b) = \mathbf{r}(d)$. By Lemma 2.3, $\mathbf{C}$ is right reversible and so there are elements $x, y \in \mathbf{C}$ such that $xa = yc$. As, $\mathbf{r}(x) = \mathbf{d}(a)$ and $\mathbf{r}(y) = \mathbf{d}(c)$ we have

$$ac^{-1} = x^{-1}xac^{-1} = x^{-1}ycc^{-1} = x^{-1}y.$$

Since $\mathbf{d}(a) = \mathbf{d}(b)$ and $\mathbf{d}(c) = \mathbf{d}(d)$ we have

$$ca^{-1} = ca^{-1}bb^{-1} = cc^{-1}db^{-1} = db^{-1}.$$

Hence $db^{-1} = ca^{-1} = y^{-1}x$. As $\mathbf{d}(x) = \mathbf{d}(y)$ and $\mathbf{r}(b) = \mathbf{r}(d)$ we have that $xb = yd$.

$(ii) \Rightarrow (iii)$. It is clear that $\mathbf{r}(a) = \mathbf{r}(c)$ and $\mathbf{r}(b) = \mathbf{r}(d)$. Let $xa = yc$ and $xb = yd$. Suppose that $ta = rc$ for all $t, r \in \mathbf{C}$. We have to show that $tb = rd$. By Lemma 2.3, $\mathbf{C}$ is right reversible and cancellative. Hence since $\mathbf{r}(y) = \mathbf{r}(r)$, it follows that $ky = hr$ for some $k, h \in \mathbf{C}$. Now,

$$kxa = kyc = hrc = hta,$$

cancelling in $\mathbf{C}$ gives that $kx = ht$. Then

$$htb = kxb = kyd = hrd,$$

again cancelling in $\mathbf{C}$ gives that $tb = rd$ as required.

$(iii) \Rightarrow (i)$. Since $\mathbf{C}$ is right reversible we have that $ta = rc$ for some $t, r \in \mathbf{C}$ so that $tb = rd$. Then

$$ac^{-1} = t^{-1}r = bd^{-1},$$

so that

$$a^{-1}b = a^{-1}bd^{-1}d = a^{-1}ac^{-1}d = c^{-1}d,$$

as required.

Lawson has deduced the following from [5]. He has called the groupoid $\mathbb{G}$ in such a theorem a *groupoid of fractions* of $\mathbf{C}$.

**Theorem 2.5. [9]** Let $\mathbf{C}$ be a right reversible cancellative category. Then $\mathbf{C}$ is a subcategory of a groupoid $\mathbb{G}$ such that the following three conditions hold:

($i$) $\mathbf{C_0} = \mathbb{G}_0$.

($ii$) Every element of $\mathbb{G}$ is of the form $a^{-1}b$ where $a, b \in \mathbf{C}$.

($iii$) $a^{-1}b = c^{-1}d$ if and only if there exist $x, y \in \mathbf{C}$ such that $xa = yc$ and $xb = yd$.

**Proof.** Our proof is basically the same as the proof given by Tobais Fritz [4] in the case of categories as a collections of objects and homomorphisms, but our presentation is slightly different as we shall use the second definition of categories.

From Lemmas 2.3 and 2.4, ($i$) and ($iii$) are clear.

To prove ($ii$) suppose that $\mathbf{C}$ is right reversible and cancellative. We aim to construct a groupoid $\mathbb{G}$ in which $\mathbf{C}$ is embedded as a left order in $\mathbb{G}$. This construction is based on ideas by Tobias Fritz [4] and Cegarra, the author and Petrich [1]. Let

$$\widetilde{\mathbb{G}} = \{(a, b) \in \mathbf{C} \times \mathbf{C} : \mathbf{d}(a) = \mathbf{d}(b)\}.$$

Define a relation $(a, b) \sim (c, d)$ on $\widetilde{\mathbb{G}}$ by

$$(a, b) \sim (c, d) \iff there\ exist\ x, y \in \mathbf{C}\ such\ that\ xa = yc\ and\ xb = yd.$$

We can represent this relation by the following diagram

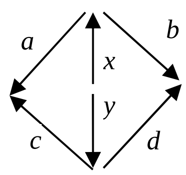


Figure 2.4

Notice that if $(a, b) \sim (c, d)$, then $\mathbf{r}(a) = \mathbf{r}(c)$ and $\mathbf{r}(b) = \mathbf{r}(d)$.

**Lemma 2.6.** The relation $\sim$ defined above is an equivalence relation.

**Proof.** It is clear that $\sim$ is symmetric and reflexive. Let

$$(a, b) \sim (c, d) \sim (p, q),$$

where $(a, b)$, $(c, d)$ and $(p, q)$ in $\widetilde{\mathbb{G}}$. Hence there exist $x, y, \bar{x}, \bar{y}, \in \mathbf{C}$ such that

$$xa = yc, xb = yd \text{ and } \bar{x}c = \bar{y}p, \bar{x}d = \bar{y}q.$$

To show that $\sim$ is transitive, we have to show that there are elements $z, \bar{z}, \in \mathbf{C}$ such that $za = \bar{z}p$ and $zb = \bar{z}q$. Since $\mathbf{C}$ is right reversible and $\mathbf{r}(y) = \mathbf{r}(\bar{x})$ there are elements $s, t \in \mathbf{C}$ such that $sy = t\bar{x}$. Hence

$$sxa = syc = t\bar{x}c = t\bar{y}p.$$

Similarly, $sxb = t\bar{y}q$ as required.

Let $[a, b]$ denote the $\sim$-equivalence class of $(a, b)$. On $\mathbb{G} = \widetilde{\mathbb{G}}/\sim$ we define a product as follows. Let $[a, b], [c, d] \in \mathbb{G}$. Their product is defined iff $\mathbf{r}(b) = \mathbf{r}(c)$. Define

$$[a, b][c, d] = \begin{cases} [xa, yd], & if\ xb = yc\ \ forsome\ x, y \in \mathbf{C}; \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

and so we have the following diagram

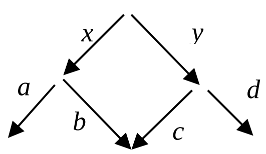


Figure 2.5

**Lemma 2.7.** The multiplication is well-defined.

**Proof.** Suppose that $[a_1, b_1] = [a_2, b_2]$ and $[c_1, d_1] = [c_2, d_2]$ are in $\mathbb{G}$. Then there are elements $x_1, x_2, y_1, y_2$ in **C** such that

$$x_1 a_1 = x_2 a_2,$$
$$x_1 b_1 = x_2 b_2,$$
$$y_1 c_1 = y_2 c_2,$$
$$y_1 d_1 = y_2 d_2.$$

Now,

$$[a_1, b_1][c_1, d_1] = [wa_1, \bar{w}d_1] \text{ and } wb_1 = \bar{w}c_1$$

for some $w, \bar{w} \in \mathbf{C}$ and

$$[a_2, b_2][c_2, d_2] = [za_2, \bar{z}d_2] \text{ and } zb_2 = \bar{z}c_2$$

for some $z, \bar{z} \in \mathbf{C}$.

It is easy to see that $[a_1, b_1][c_1, d_1]$ is defined if and only if $[a_2, b_2][c_2, d_2]$ is defined. We have to prove that $[wa_1, \bar{w}d_1] = [za_2, \bar{z}d_2]$, that is,

$$xwa_1 = yza_2 \text{ and } x\bar{w}d_1 = y\bar{z}d_2, \text{for some } x, y \in \mathbf{C}.$$

Since $wb_1$ is defined and $\mathbf{d}(a_1) = \mathbf{d}(b_1)$ we have that $wa_1$ is defined and $\mathbf{r}(a_1) = \mathbf{r}(wa_1)$. Similarly, $za_2$ is defined and $\mathbf{r}(a_2) = \mathbf{r}(za_2)$. Hence $\mathbf{r}(wa_1) = \mathbf{r}(za_2)$, by the right reversibility of **C** there are elements $x, y \in \mathbf{C}$ with $xwa_1 = yza_2$. It remains to show that $x\bar{w}d_1 = y\bar{z}d_2$. By Lemma 2.4, $xwb_1 = yzb_2$ and as $wb_1 = \bar{w}c_1$ and $zb_2 = \bar{z}c_2$ we have that $x\bar{w}c_1 = y\bar{z}c_2$ and so $x\bar{w}d_1 = y\bar{z}d_2$, again by Lemma 2.4.

**Lemma 2.8.** The multiplication is associative.

**Proof.** Let $[a, b], [c, d], [p, q] \in \mathbb{G}$ and set

$$X = ([a, b][c, d])[p, q] = [xa, yd][p, q]$$

where $xb = yc$ for some $x, y \in \mathbf{C}$ and

$$Y = [a, b]([c, d][p, q]) = [a, b][\bar{x}c, \bar{y}q]$$

where $\bar{x}d = \bar{y}p$ for some $\bar{x}, \bar{y} \in \mathbf{C}$. It is clear that X is defined if and only if Y is defined. We assume that $[a, b][c, d]$ and $[c, d][p, q]$ are defined. Then for some $x, y, \bar{x}, \bar{y} \in \mathbf{C}$ we have

$$\begin{aligned} X &= [xa, yd][p, q] \\ &= [sxa, rq] \end{aligned}$$

where $syd = rp$ for some $s, r \in \mathbf{C}$.

$$\begin{aligned} Y &= [a, b][\bar{x}c, \bar{y}q] \\ &= [\bar{s}a, \bar{r}\bar{y}q] \end{aligned}$$

where $\bar{s}b = \bar{r}\bar{x}c$ for some $\bar{s}, \bar{r} \in \mathbf{C}$. We have to show that

$$X = [sxa, rq] = [\bar{s}a, \bar{r}\bar{y}q] = Y.$$

Then by definition we need to show that

$$wsxa = \bar{w}\bar{s}a \text{ and } wrq = \bar{w}\bar{r}\bar{y}q$$

for some $w, \bar{w} \in \mathbf{C}$. By cancellativity in **C** this equivalent to $wsx = \bar{w}\bar{s}$ and $wr = \bar{w}\bar{r}\bar{y}$.

Since $xb$ and $sx$ are defined we have that $sxb$ is defined and as $\bar{s}b$ is defined so that $\boldsymbol{r}(\bar{s}b) = \boldsymbol{r}(sxb)$. By right reversibility of $\mathbf{C}$ we have that $wsxb = \bar{w}\bar{s}b$ for some $w, \bar{w} \in \mathbf{C}$. By cancellativity in $\mathbf{C}$ we get $wsx = \bar{w}\bar{s}$.

Now, since $wsxb = \bar{w}\bar{s}b, \bar{s}b = \bar{r}\bar{x}c$ and $xb = yc$ we have that $wsyc = \bar{w}\bar{r}\bar{x}c$. As $\mathbf{C}$ is cancellative we have that $wsy = \bar{w}\bar{r}\bar{x}$ so that $wsyd = \bar{w}\bar{r}\bar{x}d$, but $syd = rp$ and $\bar{x}d = \bar{y}p$ so that $wrp = \bar{w}\bar{r}\bar{y}p$. Thus $wr = \bar{w}\bar{r}\bar{y}$ as required.

For $[a, b] \in \mathbb{G}$ where $xa$ is defined in $\mathbf{C}$ for some $x \in \mathbf{C}$, it is clear that$[xa, xb] \in \mathbb{G}$ and $\mathbf{d}(x)xa = xa$ and $\mathbf{d}(x)xb = xb$. Hence we have

**Lemma 2.9.** If $[a, b], [xa, xb] \in \mathbb{G}$, then $[xa, xb] = [a, b]$ for all $x \in \mathbf{C}$ such that $xa$ is defined in $\mathbf{C}$.

**Lemma 2.10.** The identities of $\mathbb{G}$ have the form $[a, a]$ where $a \in \mathbf{C}$.

**Proof.** Suppose that $e = [a, b]$ is an identity in $\mathbb{G}$ where $a, b \in \mathbf{C}$. Let $[m, n] \in \mathbb{G}$ such that $[m, n][a, b]$ is defined and

$$[m, n][a, b] = [m, n].$$

Then $[xm, yb] = [m, n]$ for some $x, y \in \mathbf{C}$ with $xn = ya$. Hence

$$uxm = vm \text{ and } uyb = vn$$

for some $u, v \in \mathbf{C}$; cancelling in $\mathbf{C}$ gives that $ux = v$ so that $uyb = vn = uxn$. Again, by cancellativity, it follows that $xn = yb$ and as $xn = ya$ we have that $yb = xn = ya$. Using cancellativity in $\mathbf{C}$ once more we obtain $a = b$. Thus $e = [a, a]$. Similarly, if $[a, b][m, n] = [m, n]$ we have that $a = b$.

It remains to show that the identity is unique. Suppose that

$$[a, b][c, c] = [a, b][d, d] = [a, b]$$

for some identities $[c, c], [d, d] \in \mathbb{G}$. Then by definition $[xa, yc] = [\acute{x}a, \acute{y}d]$ where $xb = yc$ and $\acute{x}b = \acute{y}d$ for some $x, y, \acute{x}, \acute{y} \in \mathbf{C}$. Hence $uxa = v\acute{x}a$ and $uya = v\acute{y}d$. By definition of $\sim$ and Lemma 2.9,

$$[c, c] = [yc, yc] = [\acute{y}d, \acute{y}d] = [d, d]$$

As required. Similarly, $[c, c][a, b] = [d, d][a, a] = [a, b]$ implies that $[c, c] = [d, d]$.

Suppose that $[a, b] \in \mathbb{G}$. Then as $\mathbf{d}(a)a = \mathbf{d}(a)a$ we have

$$[a, a][a, b] = [\mathbf{d}(a)a, \mathbf{d}(a)b] = [a, b].$$

Similarly, $\mathbf{d}(b)b = \mathbf{d}(b)b$ whence

$$[a, b][b, b] = [\mathbf{d}(a)a, \mathbf{d}(a)b] = [a, b].$$

Hence $\mathbf{d}([a, b]) = [a, a]$ and $\mathbf{r}([a, b]) = [b, b]$.

By the above argument and Lemma 2.8, the following lemma is clear.

**Lemma 2.11.** $\mathbb{G}$ is a category.

If $[a, b] \in \mathbb{G}$, then it is clear that $[b, a] \in \mathbb{G}$, as $\mathbf{d}(a)b = \mathbf{d}(a)b$ we have

$$[a, b][b, a] = [\mathbf{d}(a)a, \mathbf{d}(a)a] = [a, a] = \mathbf{d}([a, b]).$$

Similarly, $[b, a][a, b] = [b, b] = \mathbf{r}([a, b])$. That is, $[b, a]$ is the inverse of $[a, b]$ in $\mathbb{G}$. Thus we have

**Lemma 2.12.** $\mathbb{G}$ is a groupoid.

**Lemma 2.13.** The mapping $\theta: \mathbf{C} \to \mathbb{G}$ defined by $a\theta = [\mathbf{d}(a), a]$ is an embedding of $\mathbf{C}$ in $\mathbb{G}$.

**Proof.** It is clear that $\theta$ is well-defined. To show that $\theta$ is one-to-one, let $[\mathbf{d}(a), a] = [\mathbf{d}(b), b]$ so that $ua = vb$ and $u\mathbf{d}(a) = v\mathbf{d}(b)$ for some $u, v \in \mathbf{C}$. Hence $a = b$.

Let $a, b \in \mathbf{C}$ such that ab is defined. We have

$$\begin{aligned} a\theta b\theta &= [\mathbf{d}(a), a][\mathbf{d}(b), b] \\ &= [u\mathbf{d}(a), vb] && \text{where } ua = v\mathbf{d}(b) \text{ for some } u, v \in \mathbf{C} \\ &= [u\mathbf{d}(a), uab] && \text{as } ua = v\mathbf{d}(b) = v \\ &= [\mathbf{d}(a), ab] && \text{by Lemma 2.9} \\ &= [\mathbf{d}(ab), ab] && \text{as } \mathbf{d}(a) = \mathbf{d}(ab) \\ &= (ab)\theta. \end{aligned}$$

Thus $\theta$ is a homomorphism.

From (*i*) we know that $\mathbf{C}_0 = \mathbb{G}_0$. Hence $\mathbf{C}$ is a left order in $\mathbb{G}$. This completes the proof of Theorem 2.5.

**Corollary 2.14.** A subcategory $\mathbf{C}$ is a left order in a groupoid $\mathbb{G}$ if and only if $\mathbf{C}$ is right reversible and cancellative.

**Proof.** If $\mathbf{C}$ is a left order in a groupoid $\mathbb{G}$, then by Lemma 2.3, $\mathbf{C}$ is right reversible and cancellative. Conversely, if $\mathbf{C}$ is right reversible and cancellative, then by (*ii*) in Theorem 2.5, $\mathbf{C}$ is a left order in a groupoid $\mathbb{G}$.

## IV. Uniqueness

In this section we show that a category $\mathbf{C}$ has, up to isomomorphism, at most one groupoid of left I-quotients.

**Theorem 3.1.** Let $\mathbf{C}$ be a left order in groupoid $\mathbb{G}$. If $\varphi$ is an embedding of $\mathbf{C}$ to a groupoid $\mathbb{T}$, then there is a unique embedding $\psi: \mathbb{G} \to \mathbb{T}$ such that $\psi|_{\mathbf{C}} = \varphi$.

**Proof.** Define $\psi: \mathbb{G} \to \mathbb{T}$ by

$$(a^{-1}b)\psi = (a\varphi)^{-1}(b\varphi)$$

$a, b, c \in \mathbf{C}$. Suppose that

$$a^{-1}b = c^{-1}d$$

so that $xa = yc$ and $xb = yd$ for some $x, y \in \mathbf{C}$, by Lemma 2.4. Hence

$$x\varphi a\varphi = y\varphi c\varphi \; and \; x\varphi b\varphi = y\varphi d\varphi$$

in $\mathbf{C}\varphi$. Thus

$$a\varphi c\varphi^{-1} = x\varphi^{-1}y\varphi = b\varphi d\varphi^{-1}$$

so that

$$a\varphi^{-1}b\varphi = c\varphi^{-1}d\varphi.$$

It follows that $\psi$ is well-defined and 1-1. It remains for us to show that $\psi$ is a homomorphism. Let $a^{-1}b, c^{-1}d \in \mathbb{G}$ where $a, b, c, d \in \mathbf{C}$. Now,

$$\begin{aligned} (a^{-1}bc^{-1}d)\psi &= ((xa)^{-1}(xa))\psi \\ &= (xa)\varphi^{-1}(yd)\varphi \\ &= a\varphi^{-1}x\varphi^{-1}y\varphi d\varphi, \end{aligned}$$

where $xb = yc$ for some $x, y \in \mathbf{C}$. We have that $x\varphi b\varphi = y\varphi c\varphi$ and so $b\varphi c\varphi^{-1} = x\varphi^{-1}y\varphi$. Hence

$$\begin{aligned} (a^{-1}bc^{-1}d)\psi &= a\varphi^{-1}x\varphi^{-1}y\varphi d\varphi \\ &= a\varphi^{-1}b\varphi \, c\varphi^{-1}d\varphi \\ &= (a^{-1}b)\psi(c^{-1}d)\psi. \end{aligned}$$

Finally, to see that $\psi$ is unique, suppose that $\theta: \mathbb{G} \to \mathbb{T}$ is an embedding with $\theta|_{\mathbf{C}} = \varphi$. Then for an element $a^{-1}b$ of $\mathbb{G}$, we have

$$(a^{-1}b)\theta = (a^{-1}\theta)(b\theta) = (a\theta)^{-1}(b\theta) = (a\varphi)^{-1}(b\varphi) = (a^{-1}b)\psi$$

so that $\theta = \psi$.

The following corollary is straightforward.

**Corollary 3.2.** If a category **C** is a left order in groupoids $\mathbb{G}$ and $\mathbb{P}$, then $\mathbb{G}$ and $\mathbb{P}$ are isomorphic by an isomorphism which restricts to the identity map on **C**.

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