Generalized Identities on Products of Fibonacci-Like and Lucas Numbers

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***Abstract*-The Fibonacci, Fibonacci-Like and Lucas sequences are shining stars in the vast array of integer sequences. They have fascinated both amateurs and professional mathematicians for centuries. Also they continue to charm us with their beauty, their abundant applications and their ubiquitous habit of occurring in totally surprising and unrelated places. The product of Fibonacci number and Lucas number is a linear function of Fibonacci numbers. In this paper, we investigated some generalized identities on products of Fibonacci-Like and Lucas numbers. Further we showed that product is commutative when same location numbers will be taken in reverse order.**

Keywords- Fibonacci-Like numbers, Lucas numbers, Binet's formula, Identities.

I. INTRODUCTION

Mathematics can be considered as underlying order of the universe, and the Fibonacci numbers are one of the most fascinating discoveries made in the mathematical world. Among numerical sequences, the Fibonacci and Lucas sequences have achieved a kind of celebrity status and have been studied extensively in number theory, applied mathematics, physics, computer science, and biology [6].

The Fibonacci sequence in each next term is sum of previous two terms with two specific initial values, is a source of many nice and interesting identities. A similar interpretation also exists for Lucas and Fibonacci-Like sequences. The Fibonacci and Lucas numbers have been studied both for their applications and mathematical beauty of rich and interesting identities that they satisfy. The Fibonacci-Like numbers which is obtained as sum of Fibonacci and Lucas numbers also possesses rich and interesting identities and properties.

The Fibonacci sequence (0, 1, 1, 2, 3, 5, 8,…) [6] is given by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}\ ,\ n \geq 2 \text{ with } F_0 = 0,\ F_1 = 1. \tag{1}$$

The Lucas sequence (2, 1, 3, 4, 7, 11, 18,…) [6] is given by the recurrence relation:

$$L_n = L_{n-1} + L_{n-2}\ ,\ n \geq 2 \text{ with } L_0 = 2,\ L_1 = 1.$$

(2)

The Fibonacci-Like sequence (2, 2, 4, 6, 10, 16, 26,…) [3] is given by the recurrence relation:

$$S_n = S_{n-1} + S_{n-2}\ ,\ n \geq 2 \text{ with } S_0 = 2,\ S_1 = 2.$$

(3)

The associated initial conditions S_0 and S_1 are sum of initial conditions of Fibonacci and Lucas sequence respectively.

i. e., $S_0 = F_0 + L_0$ and $S_1 = F_1 + L_1$.

The Binet's formula for Fibonacci sequence is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

(4)

The Binet's formula for Lucas sequence is given by

$$L_n = \alpha^n + \beta^n.$$

(5)

The Binet's formula for Fibonacci-Like sequence is given by

$$S_n = 2\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

(6)

where α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$ and $\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$ (Golden ratio) and $\beta = \frac{1-\sqrt{5}}{2} \approx -1.618$ (Another golden ratio).

Also, $\alpha + \beta = 1,\ \alpha - \beta = \sqrt{5},\ \alpha\beta = -1.$

(7)

The product of Fibonacci numbers and Lucas number is a linear function of Fibonacci numbers. In this paper we illustrated some generalized identities on products of Fibonacci-Like and Lucas numbers. Further we showed that product is commutative when same location numbers will be taken in reverse order.

II. SOME GENERALIZED IDENTITIES

There are a lot of identities of Fibonacci and Lucas numbers described in [6]. Some properties for common factors of Fibonacci and Lucas numbers are studied by Thongmoon [4, 5]. Singh *et al.* [1] investigated some generalized identities involving common factors of Fibonacci and Lucas numbers. Some identities established for even and odd Fibonacci-Like and Lucas numbers by Singh *et al.* [2].

Koshy [7] illustrated sums of Fibonacci–Pell–Jacobsthal products and Cerin [8] described sums of products of generalized Fibonacci and Lucas numbers. Now we illustrate some generalized identities on products of Fibonacci-Like and Lucas numbers.

Theorem (2.1). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k+p}L_{2k+p} = S_{4k+2p} + 2\left(-1\right)^{2k+p}.$$

Proof. By Binet's formula (5) and (6), we have

$$S_{2k+p}L_{2k+p} = 2\left(\frac{\alpha^{2k+p+1} - \beta^{2k+p+1}}{\alpha-\beta}\right)\left(\alpha^{2k+p} + \beta^{2k+p+1}\right),$$

$$= 2\left(\frac{\alpha^{4k+2p+1} - \beta^{4k+2p+1}}{\alpha-\beta}\right) + \frac{2}{\alpha-\beta}\left(\alpha\beta\right)^{2k+p}\left(\alpha-\beta\right).$$

By using Binet's formula (6) and (7), it follows that

$$S_{2k+p}L_{2k+p} = S_{4k+2p} + 2\left(-1\right)^{2k+p}.$$

The proofs of following identities are similar.

Theorem (2.2). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k-p}L_{2k-p} = S_{4k-2p} + 2\left(-1\right)^{2k-p}.$$

Theorem (2.3). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k+p}L_{2k-p} = S_{4k} + \left(-1\right)^{2k-p} S_{2p}.$$

Theorem (2.4). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k-p}L_{2k+p} = S_{4k} + \left(-1\right)^{2k-p} S_{2p-2}.$$

Theorem (2.5). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k+p}L_{2k} = S_{4k+p} + S_{p}.$$

Theorem (2.6). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k}L_{2k+p} = S_{4k+p} + S_{p-2}.$$

Theorem (2.7). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k+p+1}L_{2k} = S_{4k+p+1} + S_{p+1}.$$

Theorem (2.8). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k}L_{2k+p+1} = S_{4k+p+1} + S_{p-1}.$$

Theorem (2.9). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k+p-1}L_{2k} = S_{4k+p-1} + S_{p-1}.$$

Theorem (2.10). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k}L_{2k+p-1} = S_{4k+p-1} + S_{p-3}.$$

Theorem (2.11). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k+p}L_{2k+p+1} = S_{4k+2p+1}.$$

Theorem (2.12). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k+p-1}L_{2k+p} = S_{4k+2p-1}.$$

Theorem (2.13). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k+p+1}L_{2k+p} = S_{4k+2p+1} + 2\left(-1\right)^{2k+p}.$$

Theorem (2.14). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k+p}L_{2k+p-1} = S_{4k+2p-1} - 2\left(-1\right)^{2k+p}.$$

Theorem (2.15). For positive integers $k \geq 1,\ p \geq 0,$ prove that

$$S_{2k}L_{2k-p} = S_{4k-p} + \left(-1\right)^{2k-p} S_p.$$

We observe that product of Fibonacci-Like and Lucas numbers is commutative when same location numbers will be taken in reverse order (Theorem 2.1 and 2.2), otherwise not commutative. Also the product of Fibonacci-Like numbers and Lucas numbers is a linear function of Fibonacci-Like numbers.

III. CONCLUSION

In this paper we have illustrated some generalized identities on products of Fibonacci-Like and Lucas numbers. Further verified that product is commutative when same location numbers will be taken in reverse order. The Binet's formulae of respective sequences have been used to derive the generalized identities. The product of more numbers of special sequences can be taken in second order and can be extended same in higher orders.

IV. ACKNOWLEDGMENT

We would like to thank the anonymous referee for numerous helpful suggestions.

REFERENCES

[1] B. Singh, P. Bhadouria, O. Sikhwal, "Generalized identities involving common factors of Fibonacci and Lucas numbers", *International Journal of Algebra*, Vol. 5, No. 13, pp. 637-645, 2011.

[2] B. Singh, O. Sikhwal, S. Bhatnagar, "Some identities for even and odd Fibonacci-Like and Lucas numbers", *Proc. National Workshop-Cum-Conference on Recent Trends in Mathematics and Computing*, USA, 2011, pp 4-6.

[3] B. Singh, O. Sikhwal, S. Bhatnagar, "Fibonacci-Like sequence and its properties", , *International Journal of Contemporary Mathematical Sciences*, Vol. 5, No. 18, pp. 859-868, 2010.

[4] M. Thongmoon, "Identities for the common factors of Fibonacci and Lucas numbers", *International Mathematical Forum*, Vol. 4, no. 7, pp. 303-308, 2009.

[5] M. Thongmoon, "New identities for even and odd Fibonacci and Lucas numbers", *International Journal of Contemporary Mathematical Sciences*, Vol. 4, No. 14, pp. 71–676, 2009.

[6] T. Koshy, "Fibonacci and Lucas numbers with applications", New York, NY, USA, Wiley-Interscience, 2001.

[7] T. Koshy, "Sums of Fibonacci–Pell–Jacobsthal products", *International Journal of Mathematical Education in Science and Technology*, Vol. 44, No. 4, pp. 559-568, 2013.

[8] Z. Cerin, "Sums of products of generalized Fibonacci and Lucas numbers", *Demonstratio Mathematica*, Vol. 42, No. 2, pp. 211-218, 2009.