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On 2- and 4- Dissection of a Continued Fraction of Ramanujan

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Abstract. In this paper we give an easy proof of 2- and 4-dissection of a continued fraction of Ramanujan and its reciprocal, by using only Jacobi triple product identity.

1 Introduction

In Entries (4.3.1) and (4.3.2) of Ramanujan's Lost Notebook [2]. We find product formulas, given by Ramanujan, for the four series obtained by 2-dissection of the series for Rogers-Ramanujan continued fraction and its reciprocal. Recently Hirchhorn [10] gave a 2-dissection and 4-dissection of Rogers-Ramanujan continued fraction. This motivated us to give 2-dissection and 4-dissection of a continued fraction of Ramanujan and its reciprocal, given in [4, Entry 12, p.24]. We have specialized the parameters in [4, Entry 12, p.24], by making $q \to q^3$ and taking $a = q^{5/2}$ and $b = q^{1/2}$ to get

$$\frac{(q^4, q^8; q^{12})_\infty}{(q^2, q^{10}; q^{12})_\infty} = \frac{1}{1_+} \frac{1-q^2}{1-q^3}{}_+ \frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)}{}_+ \frac{(q^{5/2}-q^{19/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)}{}_+ \ldots, \tag{1.1}$$

where, $(a;q^k)_n = \prod_{j=1}^{n}(1-aq^{k(j-1)})$,

$(a;q^k)_\infty = \prod_{j=1}^{\infty}(1-aq^{k(j-1)})$,

and $(a;q^k)_0 = 1$.
In this paper we will consider the following continued fraction by letting $q \to q^{1/2}$ for giving the dissection.

$$\begin{aligned} P(q) &= \frac{1}{1_+} \frac{1-q}{1-q^{3/2}}{}_+ \frac{(q^{5/4}-q^{7/4})(q^{1/4}-q^{11/4})}{(1-q^{3/2})(q^3+1)}{}_+ \frac{(q^{5/4}-q^{19/4})(q^{1/4}-q^{23/4})}{(1-q^{3/2})(q^6+1)}{}_+ \ldots \\ &= \frac{(q^2,q^4;q^6)_\infty}{(q,q^5;q^6)_\infty}. \end{aligned} \tag{1.2}$$

MSC 2010: 33D15, 11A55.
Key Words: q-Hypergeometric Series, Continued Fraction, Dissection.

2 Definition of dissection

The m-dissection of the power series $P = \sum_{n=0}^{\infty} a_n q^n$ is the representation of P as $P = P_0 + P_1 + P_2 + ... + P_{m-1}$, where $P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$. The m-dissection of continued fractions are represented in terms of power series and infinite products.

3 2-dissection of the continued fraction $P(q)$

We give the 2−dissection of the continued fraction $P(q)$, using triple product identity We shall use the Jacobi Triple Product Identity [1]

$$\prod_{n=1}^{\infty}(1 + aq^{2n-1})(1 + a^{-1}q^{2n-1})(1 - q^{2n}) = \sum_{n=-\infty}^{\infty} q^{n^2} a^n. \tag{3.1}$$

Theorem 1.

$$(i) \quad \sum_{n=0}^{\infty} a_{2n} q^n = \frac{(q, q^2; q^3)_\infty(-q^4, -q^8, q^{12}; q^{12})_\infty}{(q^3; q^3)_\infty(q, q^5; q^6)_\infty},$$

$$(ii) \quad \sum_{n=0}^{\infty} a_{2n+1} q^n = \frac{(q, q^2; q^3)_\infty(-q^2, -q^{10}, q^{12}; q^{12})_\infty}{(q^3; q^3)_\infty(q, q^5; q^6)_\infty}.$$

Proof of Theorem 1.

The continued fraction

$$P(q) = \frac{1}{1} {}_{+} \frac{1-q}{1-q^{3/2}} {}_{+} \frac{(q^{5/4} - q^{7/4})(q^{1/4} - q^{11/4})}{(1-q^{3/2})(q^3+1)} {}_{+} \cdots$$

$$, = \frac{(q^2, q^4; q^6)_\infty}{(q, q^5; q^6)_\infty}.$$

Let

$$P(q) = \sum_{n=0}^{\infty} a_n q^n,$$

then

$$\sum_{n=0}^{\infty} a_n q^n = \frac{(q^2, q^4; q^6)_\infty}{(q, q^5; q^6)_\infty}$$

$$= \frac{(q^2, q^4; q^6)_\infty}{\prod_{n=1}^{\infty}(1 - q^{6n-5})(1 - q^{6n-1})}$$

$$= \frac{(q^2, q^4; q^6)_\infty \prod_{n=1}^{\infty}(1 + q^{6n-5})(1 + q^{6n-1})(1 - q^{6n})}{\prod_{n=1}^{\infty}(1 - q^{12n-10})(1 - q^{12n-2})(1 - q^{6n})}$$

$$= \frac{(q^2,q^4;q^6)_\infty \prod_{n=1}^{\infty}(1+q^{6n-5})(1+q^{6n-1})(1-q^{6n})}{(q^6;q^6)_\infty (q^2,q^{10};q^{12})_\infty}. \tag{3.2}$$

Letting $q \to q^3$, and taking $a = q^2$, in (3.1)

$$\prod_{n=1}^{\infty}(1+q^{6n-1})(1+q^{6n-5})(1-q^{6n}) = \sum_{n=-\infty}^{\infty} q^{3n^2+2n} \tag{3.3}$$

Using (3.3) in (3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_n q^n &= \frac{(q^2,q^4;q^6)_\infty}{(q^6;q^6)_\infty (q^2,q^{10};q^{12})_\infty} \sum_{n=-\infty}^{\infty} q^{3n^2+2n} \\ &= \frac{(q^2,q^4;q^6)_\infty}{(q^6;q^6)_\infty (q^2,q^{10};q^{12})_\infty} \left(\sum_{n=-\infty}^{\infty} q^{12n^2+4n} + \sum_{n=-\infty}^{\infty} q^{3(2n-1)^2+2(2n-1)} \right) \\ &= \frac{(q^2,q^4;q^6)_\infty}{(q^6;q^6)_\infty (q^2,q^{10};q^{12})_\infty} \left(\sum_{n=-\infty}^{\infty} q^{12n^2+4n} + \sum_{n=-\infty}^{\infty} q^{12n^2-8n+1} \right)\end{aligned} \tag{3.4}$$

Again letting $q \to q^{12}, a = q^4$ and $q \to q^{12}, a = q^{-8}$ in eqn.(3.1) and using it in eqn.(3.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_n q^n &= \frac{(q^2,q^4;q^6)_\infty}{(q^6;q^6)_\infty (q^2,q^{10};q^{12})_\infty} \left[(-q^8,-q^{16},q^{24};q^{24})_\infty + q(-q^4,-q^{20},q^{24};q^{24})_\infty\right] \\ &= \frac{(q^2,q^4;q^6)_\infty(-q^8,-q^{16},q^{24};q^{24})_\infty}{(q^6;q^6)_\infty (q^2,q^{10};q^{12})_\infty} + q\frac{(q^2,q^4;q^6)_\infty(-q^4,-q^{20},q^{24};q^{24})_\infty}{(q^6;q^6)_\infty (q^2,q^{10};q^{12})_\infty}.\end{aligned} \tag{3.5}$$

From equation (3.5), we have

$$\sum_{n=0}^{\infty} a_{2n} q^{2n} = \frac{(q^2,q^4;q^6)_\infty(-q^8,-q^{16},q^{24};q^{24})_\infty}{(q^6;q^6)_\infty (q^2,q^{10};q^{12})_\infty}, \tag{3.6}$$

and

$$\sum_{n=0}^{\infty} a_{2n+1} q^{2n+1} = q\frac{(q^2,q^4;q^6)_\infty(-q^4,-q^{20},q^{24};q^{24})_\infty}{(q^6;q^6)_\infty (q^2,q^{10};q^{12})_\infty}. \tag{3.7}$$

Writing q for q^2 in (3.6) and (3.7), we have

$$\sum_{n=0}^{\infty} a_{2n} q^{n} = \frac{(q,q^2;q^3)_\infty(-q^4,-q^8,q^{12};q^{12})_\infty}{(q^3;q^3)_\infty (q,q^5;q^6)_\infty}, \tag{3.8}$$

and

$$\sum_{n=0}^{\infty} a_{2n+1} q^{n} = \frac{(q,q^2;q^3)_\infty(-q^2,-q^{10},q^{12};q^{12})_\infty}{(q^3;q^3)_\infty (q,q^5;q^6)_\infty}. \tag{3.9}$$

Thus we have Theorem 1.

4 4-dissection of the continued fraction

We now give the 4-dissection of $P(q)$,
Theorem 2.

$$\sum_{n=0}^{\infty} a_{4n}q^n = \frac{(-q^2,-q^4,q^6;q^6)_\infty (q,q^2;q^3)_\infty (q^{12},q^{24},q^{30},q^{48};q^{48})_\infty}{(q^3;q^3)_\infty^2 (q^9,q^{15};q^{24})_\infty}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} a_{4n+1}q^n = \frac{(-q,-q^5,q^6;q^6)_\infty (q,q^2;q^3)_\infty (q^{12},q^{24},q^{30},q^{48};q^{48})_\infty}{(q^3;q^3)_\infty^2 (q^9,q^{15};q^{24})_\infty}, \tag{4.2}$$

$$\sum_{n=0}^{\infty} a_{4n+2}q^n = \frac{q(-q^2,-q^4,q^6;q^6)_\infty (q,q^2;q^3)_\infty (q^6,q^{24},q^{42},q^{48};q^{48})_\infty}{(q^3;q^3)_\infty^2 (q^3,q^{21};q^{24})_\infty}, \tag{4.3}$$

$$\sum_{n=0}^{\infty} a_{4n+2}q^n = \frac{q(-q,-q^5,q^6;q^6)_\infty (q,q^2;q^3)_\infty (q^6,q^{24},q^{42},q^{48};q^{48})_\infty}{(q^3;q^3)_\infty^2 (q^3,q^{21};q^{24})_\infty}. \tag{4.4}$$

Proof of Theorem 2.
The 2-dissection of $\frac{1}{(q;q)_\infty}$, Hirschhorn [10]

$$\frac{1}{(q;q)_\infty} = \frac{1}{(q^2;q^2)_\infty^2}\left(\frac{(q^{12},q^{16},q^{20},q^{32};q^{32})_\infty}{(q^6,q^{10};q^{16})_\infty} + q\frac{(q^4,q^{16},q^{28},q^{32};q^{32})_\infty}{(q^2,q^{14};q^{16})_\infty}\right) \tag{4.5}$$

or

$$\frac{1}{(q^3;q^3)_\infty} = \frac{1}{(q^6;q^6)_\infty^2}\left(\frac{(q^{36},q^{48},q^{60},q^{96};q^{96})_\infty}{(q^{18},q^{30};q^{48})_\infty} + q^3\frac{(q^{12},q^{48},q^{84},q^{96};q^{96})_\infty}{(q^6,q^{42};q^{48})_\infty}\right). \tag{4.6}$$

Using (4.6) in (3.6), we have

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{(-q^4,-q^8,q^{12};q^{12})_\infty (q^2,q^4;q^6)_\infty}{(q^6;q^6)_\infty^2} \times \left(\frac{(q^{36},q^{48},q^{60},q^{96};q^{96})_\infty}{(q^{18},q^{30};q^{48})_\infty} + q^3\frac{(q^{12},q^{48},q^{84},q^{96};q^{96})_\infty}{(q^6,q^{42};q^{48})_\infty}\right) \tag{4.7}$$

From (4.7), we have

$$\sum_{n=0}^{\infty} a_{4n}q^{2n} = \frac{(-q^4,-q^8,q^{12};q^{12})_\infty (q^2,q^4;q^6)_\infty (q^{36},q^{48},q^{60},q^{96};q^{96})_\infty}{(q^6;q^6)_\infty^2 (q^{18},q^{30};q^{48})_\infty}, \tag{4.8}$$

and

$$\sum_{n=0}^{\infty} a_{4n+2}q^{2n+1} = \frac{q^3(-q^4,-q^8,q^{12};q^{12})_\infty (q^2,q^4;q^6)_\infty (q^{12},q^{48},q^{84},q^{96};q^{96})_\infty}{(q^6;q^6)_\infty^2 (q^6,q^{42};q^{48})_\infty}. \tag{4.9}$$

Writing q for q^2, we have

$$\sum_{n=0}^{\infty} a_{4n}q^n = \frac{(-q^2,-q^4,q^6;q^6)_\infty (q,q^2;q^3)_\infty (q^{18},q^{24},q^{30},q^{48};q^{48})_\infty}{(q^3;q^3)_\infty^2 (q^9,q^{15};q^{24})_\infty}, \tag{4.10}$$

and

$$\sum_{n=0}^{\infty} a_{4n+2}q^n = \frac{q(-q^2,-q^4,q^6;q^6)_\infty(q,q^2;q^3)_\infty(q^6,q^{24},q^{42},q^{48};q^{48})_\infty}{(q^3;q^3)^2_\infty(q^3,q^{21};q^{24})_\infty}. \quad (4.11)$$

Now

$$\begin{aligned}\sum_{n=0}^{\infty} a_{2n+1}q^n &= \frac{(q,q^2;q^3)_\infty(-q^2,-q^{10},q^{12};q^{12})_\infty}{(q^3;q^3)_\infty(q,q^5;q^6)_\infty}\\ &= \frac{(q^2,q^4;q^6)_\infty(-q^2,-q^{10},q^{12};q^{12})_\infty}{(q^3;q^3)_\infty}.\end{aligned}$$

Again using (4.6) we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{2n+1}q^n &= \frac{(q^2,q^4;q^6)_\infty(-q^2,-q^{10},q^{12};q^{12})_\infty}{(q^6;q^6)^2_\infty}\\ &\times\left(\frac{(q^{36},q^{48},q^{60},q^{96};q^{96})_\infty}{(q^{18},q^{30};q^{48})_\infty}+q^3\frac{(q^{12},q^{48},q^{84},q^{96};q^{96})_\infty}{(q^6,q^{42};q^{48})_\infty}\right) \quad (4.12)\end{aligned}$$

From (4.12) we have

$$\sum_{n=0}^{\infty} a_{4n+1}q^{2n} = \frac{(-q^2,-q^{10},q^{12};q^{12})_\infty(q^2,q^4;q^6)_\infty(q^{36},q^{48},q^{60},q^{96};q^{96})_\infty}{(q^6;q^6)^2_\infty(q^{18},q^{30};q^{48})_\infty}, \quad (4.13)$$

and

$$\sum_{n=0}^{\infty} a_{4n+3}q^{2n+1} = \frac{q^3(-q^2,-q^{10},q^{12};q^{12})_\infty(q^2,q^4;q^6)_\infty(q^{12},q^{48},q^{84},q^{96};q^{96})_\infty}{(q^6;q^6)^2_\infty(q^6,q^{42};q^{48})_\infty}. \quad (4.14)$$

Writing q for q^2, we get

$$\sum_{n=0}^{\infty} a_{4n+1}q^n = \frac{(-q,-q^5,q^6;q^6)_\infty(q,q^2;q^3)_\infty(q^{18},q^{24},q^{30},q^{48};q^{48})_\infty}{(q^3;q^3)^2_\infty(q^9,q^{15};q^{24})_\infty}, \quad (4.15)$$

and

$$\sum_{n=0}^{\infty} a_{4n+3}q^n = \frac{q(-q,-q^5,q^6;q^6)_\infty(q,q^2;q^3)_\infty(q^6,q^{24},q^{42},q^{48};q^{48})_\infty}{(q^3;q^3)^2_\infty(q^3,q^{21};q^{24})_\infty}. \quad (4.16)$$

Eqns. (4.10), (4.11), (4.15) and (4.16) give Theorem 2.

5 2-dissection of the reciprocal of the continued fraction

We now give 2-dissection of $\dfrac{1}{P(q)}$,

Theorem 3.

$$\sum_{n=0}^{\infty} b_{2n} q^n = \frac{(-q^4, -q^8, q^{12}; q^{12})_\infty}{(q, q^2, q^3; q^3)_\infty}$$

$$\sum_{n=0}^{\infty} b_{2n+1} q^n = -\frac{(-q^2, -q^{10}, q^{12}; q^{12})_\infty}{(q, q^2, q^3; q^3)_\infty}$$

Let

$$\frac{1}{P(q)} = \frac{(q, q^5; q^6)_\infty}{(q^2, q^4; q^6)_\infty} = \sum_{n=0}^{\infty} b_n q^n,$$

So

$$\sum_{n=0}^{\infty} b_n q^n = \frac{(q, q^5; q^6)_\infty}{(q^2, q^4; q^6)_\infty}$$

$$= \frac{\prod_{n=1}^{\infty}(1-q^{6n-5})(1-q^{6n-1})}{(q^2, q^4; q^6)_\infty}$$

$$= \frac{\prod_{n=1}^{\infty}(1-q^{6n-5})(1-q^{6n-1})(1-q^{6n})}{(q^2, q^4, q^6; q^6)_\infty}.$$

Writing q^3 for q and $a = -q^2$ in (3.1), we have

$$\prod_{n=1}^{\infty}(1-q^{6n-1})(1-q^{6n-5})(1-q^{6n}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+2n}. \tag{5.1}$$

Using (5.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_n q^n &= \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n}}{(q^2, q^4, q^6; q^6)_\infty} \\ &= \frac{1}{(q^2, q^4, q^6; q^6)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{12n^2+4n} + \sum_{n=-\infty}^{\infty} (-1)^{2n-1} q^{3(2n-1)^2+2(2n-1)}\right) \\ &= \frac{1}{(q^2, q^4, q^6; q^6)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{12n^2+4n} - \sum_{n=-\infty}^{\infty} q^{12n^2-8n+1}\right) \qquad (5.2)\end{aligned}$$

Writing q^{12} for q, $a = q^4$ and q^{12} for q , $a = q^{-8}$ in (3.1) we have

$$\prod_{n=1}^{\infty}(1+q^{24n-8})(1+q^{24n-16})(1-q^{24n}) = \sum_{n=-\infty}^{\infty} q^{12n^2+4n}. \tag{5.3}$$

and

$$\prod_{n=1}^{\infty}(1+q^{24n-20})(1+q^{24n-4})(1-q^{24n}) = \sum_{n=-\infty}^{\infty} q^{12n^2-8n}. \tag{5.4}$$

By (5.3) and (5.4), we have

$$\sum_{n=0}^{\infty} b_n q^n = \frac{1}{(q^2,q^4;q^6)_\infty}\left[(-q^8,-q^{16},q^{24};q^{24})_\infty + q(-q^4,-q^{20},q^{24};q^{24})_\infty\right]$$
$$= \frac{(-q^8,-q^{16},q^{24};q^{24})_\infty}{(q^2,q^4,q^6;q^6)_\infty} + q\frac{(-q^4,-q^{20},q^{24};q^{24})_\infty}{(q^2,q^4,q^6;q^6)_\infty} \tag{5.5}$$

From (5.5), we have

$$\sum_{n=0}^{\infty} b_{2n} q^{2n} = \frac{(-q^8,-q^{16},q^{24};q^{24})_\infty}{(q^2,q^4,q^6;q^6)_\infty}, \tag{5.6}$$

and

$$\sum_{n=0}^{\infty} b_{2n+1} q^{2n+1} = -q\frac{(-q^4,-q^{20},q^{24};q^{24})_\infty}{(q^2,q^4,q^6;q^6)_\infty}. \tag{5.7}$$

Writing q for q^2 in (5.6) and (5.7), we have

$$\sum_{n=0}^{\infty} b_{2n} q^{n} = \frac{(-q^4,-q^{8},q^{12};q^{12})_\infty}{(q,q^2,q^3;q^3)_\infty}, \tag{5.8}$$

and

$$\sum_{n=0}^{\infty} b_{2n+1} q^{n} = -\frac{(-q^2,-q^{10},q^{12};q^{12})_\infty}{(q,q^2,q^3;q^3)_\infty}. \tag{5.9}$$

Thus we have the Theorem.

6 4-dissection of the reciprocal of the continued fraction

We now give 4-dissection of $\dfrac{1}{P(q)}$,

Theorem 4.

$$\sum_{n=0}^{\infty} b_{4n} q^{n} = \frac{(-q^2,-q^4,q^6;q^6)_\infty(q^6,q^8,q^{10},q^{16};q^{16})_\infty}{(q;q)_\infty^2(q^3,q^5;q^8)_\infty}, \tag{6.1}$$

$$\sum_{n=0}^{\infty} b_{4n+1} q^{n} = -\frac{(-q,-q^5,q^6;q^6)_\infty(q^6,q^8,q^{10},q^{16};q^{16})_\infty}{(q;q)_\infty^2(q^3,q^5;q^8)_\infty}, \tag{6.2}$$

$$\sum_{n=0}^{\infty} b_{4n+2} q^{n} = \frac{(-q^2,-q^4,q^6;q^6)_\infty(q^2,q^8,q^{14},q^{16};q^{16})_\infty}{(q;q)_\infty^2(q,q^7;q^8)_\infty}, \tag{6.3}$$

$$\sum_{n=0}^{\infty} b_{4n+3} q^{n} = -\frac{(-q,-q^5,q^6;q^6)_\infty(q^2,q^8,q^{14},q^{16};q^{16})_\infty}{(q;q)_\infty^2(q,q^7;q^8)_\infty}. \tag{6.4}$$

Proof of Theorem 4.
Equation (5.8)

$$\sum_{n=0}^{\infty} b_{2n} q^{n} = \frac{(-q^4,-q^{8},q^{12};q^{12})_\infty}{(q,q^2,q^3;q^3)_\infty} = \frac{(-q^4,-q^{8},q^{12};q^{12})_\infty}{(q;q)_\infty}.$$

Again we will use

$$\frac{1}{(q;q)_\infty} = \frac{1}{(q^2;q^2)_\infty^2}\left(\frac{(q^{12},q^{16},q^{20},q^{32};q^{32})_\infty}{(q^6,q^{10};q^{16})_\infty} + q\frac{(q^4,q^{16},q^{28},q^{32};q^{32})_\infty}{(q^2,q^{14};q^{16})_\infty}\right), \tag{6.5}$$

So

$$\sum_{n=0}^{\infty} b_{2n}q^n = \frac{(-q^4,-q^8,q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^2}\frac{(q^{12},q^{16},q^{20},q^{32};q^{32})_\infty}{(q^6,q^{10};q^{16})_\infty} + q\frac{(-q^4,-q^8,q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^2}\frac{(q^4,q^{16},q^{28},q^{32};q^{32})_\infty}{(q^2,q^{14};q^{16})_\infty} \tag{6.6}$$

Hence by (6.6)

$$\sum_{n=0}^{\infty} b_{4n}q^{2n} = \frac{(-q^4,-q^8,q^{12};q^{12})_\infty(q^{12},q^{16},q^{20},q^{32};q^{32})_\infty}{(q^2;q^2)_\infty^2(q^6,q^{10};q^{16})_\infty}, \tag{6.7}$$

and

$$\sum_{n=0}^{\infty} b_{4n+2}q^{2n+1} = \frac{q(-q^4,-q^8,q^{12};q^{12})_\infty(q^4,q^{16},q^{28},q^{32};q^{32})_\infty}{(q^2;q^2)_\infty^2(q^2,q^{14};q^{16})_\infty} \tag{6.8}$$

Writing q for q^2, we have

$$\sum_{n=0}^{\infty} b_{4n}q^{n} = \frac{(-q^2,-q^4,q^{6};q^{6})_\infty(q^{6},q^{8},q^{10},q^{16};q^{16})_\infty}{(q;q)_\infty^2(q^3,q^{5};q^{8})_\infty}, \tag{6.9}$$

and

$$\sum_{n=0}^{\infty} b_{4n+2}q^{n} = \frac{(-q^2,-q^4,q^{6};q^{6})_\infty(q^{2},q^{8},q^{14},q^{16};q^{16})_\infty}{(q;q)_\infty^2(q,q^{7};q^{8})_\infty}. \tag{6.10}$$

Also form

$$\sum_{n=0}^{\infty} b_{2n+1}q^{n} = -\frac{(-q^2,-q^{10},q^{12};q^{12})_\infty}{(q,q^2,q^3;q^3)_\infty} = \frac{(-q^2,-q^{10},q^{12};q^{12})_\infty}{(q;q)_\infty},$$

and using (4.5) , we have

$$\sum_{n=0}^{\infty} b_{2n+1}q^{n} = -\frac{(-q^2,-q^{10},q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^2} \times\left(\frac{(q^{12},q^{16},q^{20},q^{32};q^{32})_\infty}{(q^6,q^{10};q^{16})_\infty} + q\frac{(q^4,q^{16},q^{28},q^{32};q^{32})_\infty}{(q^2,q^{14};q^{16})_\infty}\right). \tag{6.11}$$

Equation (6.11) gives

$$\sum_{n=0}^{\infty} b_{4n+1}q^{2n} = -\frac{(-q^2,-q^{10},q^{12};q^{12})_\infty(q^{12},q^{16},q^{20},q^{32};q^{32})_\infty}{(q^2;q^2)_\infty^2(q^6,q^{10};q^{16})_\infty}, \tag{6.12}$$

and

$$\sum_{n=0}^{\infty} b_{4n+3}q^{2n+1} = -\frac{q(-q^2,-q^{10},q^{12};q^{12})_\infty(q^4,q^{16},q^{28},q^{32};q^{32})_\infty}{(q^2;q^2)_\infty^2(q^2,q^{14};q^{16})_\infty}. \tag{6.13}$$

Writing q for q^2, we get

$$\sum_{n=0}^{\infty} b_{4n+1}q^n = -\frac{(-q,-q^5,q^6;q^6)_\infty(q^6,q^8,q^{10},q^{16};q^{16})_\infty}{(q;q)_\infty^2(q^3,q^5;q^8)_\infty}, \tag{6.14}$$

and

$$\sum_{n=0}^{\infty} b_{4n+3}q^n = -\frac{(-q,-q^5,q^6;q^6)_\infty(q^2,q^8,q^{14},q^{16};q^{16})_\infty}{(q;q)_\infty^2(q,q^7;q^8)_\infty}. \tag{6.15}$$

By eqns. (6.9), (6.10), (6.14) and (6.15) we have Theorem 4.

Conclusion; From (3.4) we can write the 2-dissection as an expansion of q-series and then combinatorial interpretation can be given.

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