

Solutions to Fuzzy Differential Equations using Pentagonal Intuitionistic Fuzzy Numbers

Mohammed Shapique. A, I.F.E.T College of Engineering, Tamilnadu, India, shapique@gmail.com
Jesuraj. C, I.F.E.T College of Engineering, Tamilnadu, India, jesu2112@gmail.com

Abstract : This paper proposes a method to solve first order differential equation considering the initial condition as pentagonal intuitionistic fuzzy numbers(PIFNs). We described homogeneous first order ordinary differential equation (ODE) in fuzzy intuitionistic environment. The fuzzy intuitionistic measures both membership and non-membership function which is the main advantage of this paper .The solution is obtained and it is expressed in PIFNs. This is exemplified by two numerical examples. To illustrate the validity of the proposed paper we have solved two Numerical examples and graphs are plotted.

Keywords - Intuitionistic fuzzy set, Fuzzy differential equation, intuitionistic fuzzy number, pentagonal intuitionistic fuzzy number.

I. INTRODUCTION

Many real time situations are not crisp and they are uncertain in nature, for example predicting the performance measure of the communication systems, mathematical modeling for the engineering problems, biological problems and computer systems. The parameter involved in such mathematical models is not precise as a result it leads to uncertainties and therefore it is vital to represent such uncertainties in linguistic characterization which can be obtained through fuzzy numbers

The fuzzy set theory was first introduced by Zadeh [1] in 1965 as an extension of the classical notion of set which deals with degree of membership of elements in a set, and then Krasimir Atanassov generalized and introduced intuitionistic concepts in fuzzy environment [2-8] which deals with membership, non –membership and indeterminacy which has established intensifying attention since after introduction. The arithmetic operations on the intuitionistic fuzzy numbers discussed by Chang and Zadeh [9], Later Fuzzy concepts are applied in differential equations. The concept of differential equation in fuzzy environment was initiated by Kaleva [10][11] and by Seikkala [12] the fuzzy derivative was introduced by Chang and Zadeh and was followed by Dubois and Prade[13]. Further it was also contributed by Puri and Ralessec [14] and Goetschel and voxman [15].

Many papers were published related to solving fuzzy differential equation by considering initial condition as fuzzy numbers Buckley, Feuring, Hayashi [16-17] used triangular fuzzy number as initial number to solve differential equation, Duraisamy and Usha [18] used Trapezoidal fuzzy number as initial value, Bede et al[19] used LR type fuzzy number, Sankar Prasad mondal and Tapan Kumar roy[20-23] by Laplace transform and by triangular intuitionistic fuzzy number. Arithmetic operation on Pentagonal fuzzy number was discussed by Bongju Lee and Yong Sik Yun [24] and Arithmetic operation on Pentagonal fuzzy number Intuitionistic Pentagonal fuzzy number discussed by Ponnivalavan and Pathinathan[25].

Fuzzy differential equation is also widely applied in biological models, engineering models, agricultural models, inventory models and many other scientific models [26-32]

In this paper we have introduced pentagonal intuitionistic fuzzy numbers (PIFNs) which will be useful to characterize the linguistic parameter. The commonly used fuzzy numbers (i.e) the triangular fuzzy have only three values one defining a single value estimate and other two represents widest possible interval for the parameter. PIFNs have five numbers where the uncertainty involved in mathematical models can be characterized. The PIFNs measures both membership and non-membership function which is the main advantage of this paper.

The arrangement of the paper is as follows. In the first and second section we discussed definitions, properties related to fuzzy numbers and pentagonal intuitionistic fuzzy numbers. In the third section we have discussed the solution procedure of solving differential equation considering initial conditions as PIFNs, in the fourth section an example problem were solved. In the fifth section we have solved a real time example using PIFNs and we concluded our work in the sixth section

II. PRELIMINARIES

Definition 2.1 Fuzzy set

A fuzzy set is characterized by a membership function mapping of a domain space (i.e) mapping between universe of discourse X to the unit interval $[0, 1]$ given by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) / x \in X\}$ Here $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ is called the degree of membership function of the fuzzy set A .

Definition 2.2 Normal fuzzy set

A fuzzy set A of the universe of discourse X is called a normal fuzzy set if there exists at least one $x \in X$ such that $\mu_A(x) = 1$

Definition 2.3 Height of a fuzzy set

The largest membership grade obtained by any element in the fuzzy set and it is given by $h(\tilde{A}) = \sup \mu_{\tilde{A}}(x)$

Definition 2.4 Convex fuzzy set

A fuzzy set $\tilde{A} = \{x, \mu_{\tilde{A}}(x) / x \in X\}$ is said to be convex if and only if for any $x_1, x_2 \in X$, the membership function of A satisfies the condition $\mu_{\tilde{A}}\{\lambda x_1 + (1-\lambda)x_2\} \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$, $\lambda \in [0, 1]$

Definition 2.5 Intuitionistic fuzzy set

An Intuitionistic fuzzy set \tilde{A} in X is given by $\tilde{A} = \{x, \mu_{\tilde{A}}(x), \vartheta_{\tilde{A}}(x) / x \in X\}$ where $\mu_{\tilde{A}} : X \rightarrow [0, 1]$, $\vartheta_{\tilde{A}} : X \rightarrow [0, 1]$ and $\mu_{\tilde{A}}(x) + \vartheta_{\tilde{A}}(x) \leq 1$ for each x , the numbers $\mu_{\tilde{A}}(x)$ and $\vartheta_{\tilde{A}}(x)$ are the degree of membership of x to \tilde{A} respectively

(α, β) -Cuts: \tilde{A} set of (α, β) -Cut, generated by an IFS A , where $(\alpha, \beta) \in [0, 1]$ are fixed number such that $\alpha + \beta \leq 1$ is defined as $A_{\alpha, \beta} = \{x, \mu_{\tilde{A}^i}(x), \vartheta_{\tilde{A}^i}(x) / x \in X, \mu_{\tilde{A}^i}(x) \geq \alpha, \vartheta_{\tilde{A}^i}(x) \leq \beta, \alpha, \beta \in [0, 1]\}$

Definition 2.6 Intuitionistic fuzzy number

An IFS $\tilde{A}^i = \{x, \mu_{\tilde{A}^i}(x), \vartheta_{\tilde{A}^i}(x) / x \in R\}$ of the real line is called an IFN if

a) \tilde{A}^i is normal

b) \tilde{A} is convex, i.e, its membership function μ is fuzzy convex and its non-membership function \mathcal{G} is fuzzy concave

c) $\mu_{\tilde{A}^i}, \mathcal{G}_{\tilde{A}^i}$ are the upper and lower semi continuous respectively.

d) $\text{Supp } A = \{x \in X / \mathcal{G}_{\tilde{A}^i}(x) < 1\}$ is bounded

Definition 2.7 *Pentagonal intuitionistic Fuzzy number*

A PIFN \tilde{A}_p^i is a subset of IFN in R with following membership function and non-membership function as follows:

$$\mu_{\tilde{A}_p^i}(x) = \begin{cases} \frac{x-a_1}{2(a_2-a_1)}, & a_1 \leq x \leq a_2 \\ 1 + \frac{x-a_3}{2(a_3-a_2)}, & a_2 \leq x \leq a_3 \\ 1 + \frac{a_3-x}{2(a_3-a_2)}, & a_3 \leq x \leq a_4 \\ \frac{a_5-x}{2(a_5-a_4)}, & a_4 \leq x < a_5 \\ 0, & x < a_1, a_5 \leq x \end{cases} \quad \mathcal{G}_{\tilde{A}_p^i}(x) = \begin{cases} 1 + \frac{x-b_1}{2(b_1-a_2)}, & b_1 \leq x \leq a_2 \\ \frac{1}{2} + \frac{x-a_2}{2(a_2-a_3)}, & a_2 \leq x \leq a_3 \\ \frac{x-a_3}{2(a_4-a_3)}, & a_3 \leq x \leq a_4 \\ \frac{1}{2} + \frac{x-a_4}{2(b_5-a_4)}, & a_4 \leq x \leq b_5 \\ 1, & x < b_1, b_5 < x \end{cases}$$

Where $b_1 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq b_5$

PIFN is denoted by $A_{PIFN}^i = (a_1, a_2, a_3, a_4, a_5; b_1, a_2, a_3, a_4, b_5)$

Here the membership function $\mu_{\tilde{A}^i}(x)$ increases steadily with constant rate for $x \in [a_1, a_2]$ and decrease steadily for $x \in [a_3, a_5]$ but the non- membership function decreases steadily for $x \in [b_1, a_3]$ and increases steadily for $x \in [a_4, b_5]$

Definition 2.8: *Arithmetic operation of IFN based on (α, β) -cuts*

If \tilde{A}_p^i is a PIFN, then (α, β) -cuts is given by

$$A_{\alpha, \beta} = \begin{cases} [A_1(\alpha), A_4(\alpha)] \text{ for } \alpha \in [0, 0.5] \\ [A_2(\alpha), A_3(\alpha)] \text{ for } \alpha \in [0.5, 1] \\ [B_2(\beta), B_3(\beta)] \text{ for } \beta \in [0, 0.5] \\ [B_1(\beta), B_4(\beta)] \text{ for } \beta \in [0.5, 1] \end{cases}$$

Here (i) $\frac{dA_1(\alpha)}{d\alpha} > 0, \frac{dA_4(\alpha)}{d\alpha} < 0, \forall \alpha \in [0, 0.5]$

(ii) $\frac{dA_2(\alpha)}{d\alpha} > 0, \frac{dA_3(\alpha)}{d\alpha} < 0, \forall \alpha \in [0.5, 1]$

(iii) $\frac{dB_1(\beta)}{d\beta} < 0, \frac{dB_4(\beta)}{d\beta} > 0, \forall \beta \in [0.5, 1]$

(iv) $\frac{dB_2(\beta)}{d\beta} < 0, \frac{dB_3(\beta)}{d\beta} > 0, \forall \beta \in [0, 0.5]$

With the condition $\alpha + \beta \leq 1$

It is expressed as $\tilde{A}_{\alpha,\beta} = \{[A_1(\alpha), A_2(\alpha), A_3(\alpha), A_4(\alpha)]; [B_1(\beta), B_2(\beta), B_3(\beta), B_4(\beta)]\}, \alpha + \beta \leq 1, \alpha, \beta \in [0, 1]$

$$\text{Hence } A_{\alpha,\beta} = \begin{cases} [2\alpha(a_2 - a_1) + a_1, 2\alpha(a_4 - a_5) + a_5], & \alpha \in [0, 0.5] \\ [2\alpha(a_3 - a_2) - a_3 + 2a_2, 2\alpha(a_3 - a_4) + 2a_4 - a_3], & \alpha \in [0.5, 1] \\ [2\beta(a_2 - a_3) + a_3, 2\beta(a_4 - a_3) + a_3], & \beta \in [0, 0.5] \\ [2\beta(b_1 - a_2) - b_1 + 2a_2, 2\beta(b_5 - a_4) + 2a_4 - b_5], & \beta \in [0.5, 1] \end{cases}$$

Property 2.1

- (a) If $\tilde{A}_p^i = (a_1, a_2, a_3, a_4, a_5; b_1, a_2, a_3, a_4, b_5)$ is a IPFS and if $y = ka (k > 0)$ then $\tilde{y}^i = k\tilde{A}_p^i$ is also a IPFS given by $(ka_1, ka_2, ka_3, ka_4, ka_5; kb_1, ka_2, ka_3, ka_4, kb_5)$
- (b) If $\tilde{A}_p^i = (a_1, a_2, a_3, a_4, a_5; b_1, a_2, a_3, a_4, b_5)$ is a IPFS and if $y = ka (k < 0)$ then $\tilde{y}^i = k\tilde{A}_p^i$ is also a IPFS given by $(ka_5, ka_4, ka_3, ka_2, ka_1; kb_5, ka_4, ka_3, ka_2, kb_1)$

Property 2.2 Addition:

If $\tilde{A}_p^i = (a_1, a_2, a_3, a_4, a_5; b_1, a_2, a_3, a_4, b_5)$ and $\tilde{B}_p^i = (c_1, c_2, c_3, c_4, c_5; d_1, c_2, c_3, c_4, d_5)$ are any two IPFS then $\tilde{A}^i \oplus \tilde{B}^i$ is also IPFS and it is given by $\tilde{A}^i \oplus \tilde{B}^i = (a_1 + c_1, a_2 + c_2, a_3 + c_3, a_4 + c_4, a_5 + c_5; b_1 + d_1, a_2 + c_2, a_3 + c_3, a_4 + c_4, b_5 + d_5)$

Property 2.3 Subtraction

If $\tilde{A}_p^i = (a_1, a_2, a_3, a_4, a_5; b_1, a_2, a_3, a_4, b_5)$ and $\tilde{B}^i = (c_1, c_2, c_3, c_4, c_5; d_1, c_2, c_3, c_4, d_5)$ are any two IPFS then $\tilde{A}_p^i(-)\tilde{B}_p^i$ is also IPFS and it is given by

$$\tilde{A}_p^i(-)\tilde{B}_p^i = (a_1 - c_5, a_2 - c_4, a_3 - c_3, a_4 - c_2, a_5 - c_1; b_1 - d_5, a_2 - c_4, a_3 - c_3, a_4 - c_2, b_5 - d_1)$$

Property 2.4 Multiplication

If $\tilde{A}_p^i = (a_1, a_2, a_3, a_4, a_5; b_1, a_2, a_3, a_4, b_5)$ and $\tilde{B}_p^i = (c_1, c_2, c_3, c_4, c_5; d_1, c_2, c_3, c_4, d_5)$ are any two IPFS then $\tilde{A}_p^i \otimes \tilde{B}_p^i$ is also IPFS and it is given by $\tilde{A}_p^i \otimes \tilde{B}_p^i = (a_1 c_1, a_2 c_2, a_3 c_3, a_4 c_4, a_5 c_5; b_1 d_1, a_2 c_2, a_3 c_3, a_4 c_4, b_5 d_5)$

Property 2.5 Division

If $\tilde{A}_p^i = (a_1, a_2, a_3, a_4, a_5; b_1, a_2, a_3, a_4, b_5)$ and $\tilde{B}_p^i = (c_1, c_2, c_3, c_4, c_5; d_1, c_2, c_3, c_4, d_5)$ are any two IPFS then $\tilde{A}^i / \tilde{B}^i$ is also IPFS and it is given by $\tilde{A}^i / \tilde{B}^i = (a_1 / c_5, a_2 / c_4, a_3 / c_3, a_4 / c_2, a_5 / c_1; b_1 / d_5, a_2 / c_4, a_3 / c_3, a_4 / c_2, b_5 / d_1)$

Definition 2.9

If $[x_1(t, \alpha), x_2(t, \alpha), x_3(t, \alpha), x_4(t, \alpha); x'_1(t, \beta), x'_2(t, \beta), x'_3(t, \beta), x'_4(t, \beta)]$ be the solution of the intuitionistic fuzzy differential equation then the solution is written as PIFN as follows

$$(x_1(t, \alpha = 0), x_2(t, \alpha = 0.5), x_2(t, \alpha = 1), x_3(t, \alpha = 0.5), x_4(t, \alpha = 1); x'_1(t, \beta = 1), x'_2(t, \beta = 0.5), x'_2(t, \beta = 0), x'_3(t, \beta = 0.5), x'_4(t, \beta = 1))$$

Definition 2.10 Strong and weak solution intuitionistic fuzzy differential equation

Consider a first order homogeneous linear intuitionistic fuzzy ordinary differential equation $\frac{dx}{dt} = kx, x(t_0) = x_0$ with

x_0 as PIFN. The solution of the above differential equation be $\tilde{x}(t)$ and its (α, β) - cuts given by

$$(i) \frac{dx_1(\alpha)}{d\alpha} > 0, \frac{dx_4(\alpha)}{d\alpha} < 0, \forall \alpha \in [0, 0.5], x_1(t, 0.5) \leq x_4(t, 0.5)$$

$$(ii) \frac{dx_2(\alpha)}{d\alpha} > 0, \frac{dx_3(\alpha)}{d\alpha} < 0, \forall \alpha \in [0.5, 1], x_2(t, 1) \leq x_3(t, 1)$$

$$(iii) \frac{dx'_1(\beta)}{d\beta} < 0, \frac{dx'_4(\beta)}{d\beta} > 0, \forall \beta \in [0.5, 1], x'_1(t, 0) \leq x'_4(t, 0)$$

$$(iii) \frac{dx'_2(\beta)}{d\beta} < 0, \frac{dx'_3(\beta)}{d\beta} > 0, \forall \beta \in [0, 0.5], x'_2(t, 0) \leq x'_3(t, 0)$$

Otherwise the solution is a weak solution

Definition 2.11 [33]

Let $f : (a, b) \rightarrow E$ and $x_0 \in (a, b)$. we say that f is strongly generalized differential at x_0 (Bede-Gal differential) if there exists an element $f'(x_0) \in E$, such that

(i) $\forall h > 0$ Sufficiently small, $\exists f(x_0 + h)^{-h} f(x_0), \exists f(x_0)^{-h} f(x_0 - h)$ and the limits is given by

$$\lim_{h \searrow 0} \frac{f(x_0 + h)^{-h} f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0)^{-h} f(x_0 - h)}{h} = f'(x_0) \quad (\text{or})$$

(ii) $\forall h > 0$ Sufficiently small, $\exists f(x_0)^{-h} f(x_0 + h), \exists f(x_0 - h)^{-h} f(x_0)$ and the limits is given by

$$\lim_{h \searrow 0} \frac{f(x_0)^{-h} f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - h)^{-h} f(x_0)}{-h} = f'(x_0) \quad (\text{or})$$

(iii) $\forall h > 0$ Sufficiently small, $\exists f(x_0 + h)^{-h} f(x_0), \exists f(x_0 - h)^{-h} f(x_0)$ and the limits is given by

$$\lim_{h \searrow 0} \frac{f(x_0 + h)^{-h} f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h)^{-h} f(x_0)}{-h} = f'(x_0) \quad (\text{or})$$

(iv) $\forall h > 0$ Sufficiently small, $\exists f(x_0)^{-h} f(x_0 + h), \exists f(x_0)^{-h} f(x_0 - h)$ and the limits is given by

$$\lim_{h \searrow 0} \frac{f(x_0)^{-h} f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0)^{-h} f(x_0 - h)}{h} = f'(x_0)$$

Definition 2.12 [34]

Let $f : (a, b) \rightarrow E$ be a function and let $f(t) = (\underline{f}(t, r), \overline{f}(t, r))$ for $\forall r \in [0, 1]$ then

1. If f is (i) differentiable, then $\underline{f}(t, r)$ & $\overline{f}(t, r)$ are differentiable function and $f'(t) = (\underline{f}'(t, r), \overline{f}'(t, r))$
2. If f is (ii) differentiable, then $\underline{f}(t, r)$ & $\overline{f}(t, r)$ are differentiable function and $f'(t) = (\overline{f}'(t, r), \underline{f}'(t, r))$

III. DIFFERENTIAL EQUATION WITH INITIAL VALUE AS PIFN

Consider the first order homogeneous linear intuitionistic fuzzy ordinary differential equation with initial condition

$$\text{as PIFN } \frac{dx}{dt} = kx, x(0) = \tilde{A}^i = (a_1, a_2, a_3, a_4, a_5; b_1, a_2, a_3, a_4, b_5)$$

Case (i): if $k > 0$, Let $k=m$

Taking (α, β) - cuts of the above equation we get

$$\frac{d}{dt} \left[(x_1(t, \alpha), x_2(t, \alpha), x_3(t, \alpha), x_4(t, \alpha)); (x'_1(t, \beta), x'_2(t, \beta), x'_3(t, \beta), x'_4(t, \beta))) \right] = m$$

$$\left[(x_1(t, \alpha), x_2(t, \alpha), x_3(t, \alpha), x_4(t, \alpha)); (x'_1(t, \beta), x'_2(t, \beta), x'_3(t, \beta), x'_4(t, \beta))) \right]$$

The initial condition is given by

$$x(t_0; \alpha, \beta) = [(a_1(\alpha), a_2(\alpha), a_3(\alpha), a_4(\alpha)); (a'_1(\alpha), a'_2(\alpha), a'_3(\alpha), a'_4(\alpha))]$$

$$\frac{d}{dt} x_1(t, \alpha) = mx_1(t, \alpha), \frac{d}{dt} x_2(t, \alpha) = mx_2(t, \alpha), \frac{d}{dt} x_3(t, \alpha) = mx_3(t, \alpha), \frac{d}{dt} x_4(t, \alpha) = mx_4(t, \alpha)$$

$$\frac{d}{dt} x'_1(t, \beta) = mx_1(t, \beta), \frac{d}{dt} x'_2(t, \beta) = mx_2(t, \beta), \frac{d}{dt} x'_3(t, \beta) = mx_3(t, \beta), \frac{d}{dt} x'_4(t, \beta) = mx_4(t, \beta)$$

With the initial condition

$$x_1(t_0, \alpha) = a_1(\alpha), x_2(t_0, \alpha) = a_2(\alpha), x_3(t_0, \alpha) = a_3(\alpha), x_4(t_0, \alpha) = a_4(\alpha)$$

$$x'_1(t_0, \beta) = a'_1(\beta), x'_2(t_0, \beta) = a'_2(\beta), x'_3(t_0, \beta) = a'_3(\beta), x'_4(t_0, \beta) = a'_4(\beta)$$

The solution of the above differential equation is given by

$$x_1(t, \alpha) = a_1(\alpha) e^{m(t-t_0)}, x_2(t, \alpha) = a_2(\alpha) e^{m(t-t_0)}, x_3(t, \alpha) = a_3(\alpha) e^{m(t-t_0)}, x_4(t, \alpha) = a_4(\alpha) e^{m(t-t_0)}$$

$$x'_1(t, \beta) = a'_1(\beta) e^{m(t-t_0)}, x'_2(t, \beta) = a'_2(\beta) e^{m(t-t_0)}, x'_3(t, \beta) = a'_3(\beta) e^{m(t-t_0)}, x'_4(t, \beta) = a'_4(\beta) e^{m(t-t_0)}$$

Case (ii) Let $k = -m$

Taking (α, β) - cuts of the above equation we get

$$\frac{d}{dt} \left[(x_1(t, \alpha), x_2(t, \alpha), x_3(t, \alpha), x_4(t, \alpha)); (x'_1(t, \beta), x'_2(t, \beta), x'_3(t, \beta), x'_4(t, \beta))) \right] = -m$$

$$\left[(x_1(t, \alpha), x_2(t, \alpha), x_3(t, \alpha), x_4(t, \alpha)); (x'_1(t, \beta), x'_2(t, \beta), x'_3(t, \beta), x'_4(t, \beta))) \right]$$

The initial condition is given by $x(t_0; \alpha, \beta) = [(a_1(\alpha), a_2(\alpha), a_3(\alpha), a_4(\alpha)); (a'_1(\alpha), a'_2(\alpha), a'_3(\alpha), a'_4(\alpha))]$

$$\frac{d}{dt} x_1(t, \alpha) = -mx_1(t, \alpha), \frac{d}{dt} x_2(t, \alpha) = -mx_2(t, \alpha), \frac{d}{dt} x_3(t, \alpha) = -mx_3(t, \alpha), \frac{d}{dt} x_4(t, \alpha) = -mx_4(t, \alpha)$$

$$\frac{d}{dt} x'_1(t, \beta) = -mx_1(t, \beta), \frac{d}{dt} x'_2(t, \beta) = -mx_2(t, \beta), \frac{d}{dt} x'_3(t, \beta) = -mx_3(t, \beta), \frac{d}{dt} x'_4(t, \beta) = -mx_4(t, \beta)$$

With the initial condition

$$x_1(t_0, \alpha) = a_1(\alpha), x_2(t_0, \alpha) = a_2(\alpha), x_3(t_0, \alpha) = a_3(\alpha), x_4(t_0, \alpha) = a_4(\alpha)$$

$$x'_1(t_0, \beta) = a'_1(\beta), x'_2(t_0, \beta) = a'_2(\beta), x'_3(t_0, \beta) = a'_3(\beta), x'_4(t_0, \beta) = a'_4(\beta)$$

The solution of the above differential equation is given by

$$x_1(t, \alpha) = \left(\frac{a_1(\alpha) + a_4(\alpha)}{2} \right) e^{-m(t-t_0)} + \left(\frac{a_1(\alpha) - a_4(\alpha)}{2} \right) e^{m(t-t_0)}$$

$$x_2(t, \alpha) = \left(\frac{a_2(\alpha) + a_3(\alpha)}{2} \right) e^{-m(t-t_0)} + \left(\frac{a_2(\alpha) - a_3(\alpha)}{2} \right) e^{m(t-t_0)}$$

$$x_3(t, \alpha) = \left(\frac{a_2(\alpha) + a_3(\alpha)}{2} \right) e^{-m(t-t_0)} - \left(\frac{a_2(\alpha) - a_3(\alpha)}{2} \right) e^{m(t-t_0)}$$

$$x_4(t, \alpha) = \left(\frac{a_1(\alpha) + a_4(\alpha)}{2} \right) e^{-m(t-t_0)} - \left(\frac{a_1(\alpha) - a_4(\alpha)}{2} \right) e^{m(t-t_0)}$$

$$x'_1(t, \beta) = \left(\frac{a'_1(\beta) + a'_4(\beta)}{2} \right) e^{-m(t-t_0)} + \left(\frac{a'_1(\beta) - a'_4(\beta)}{2} \right) e^{m(t-t_0)}$$

$$x'_2(t, \beta) = \left(\frac{a'_2(\beta) + a'_3(\beta)}{2} \right) e^{-m(t-t_0)} + \left(\frac{a'_2(\beta) - a'_3(\beta)}{2} \right) e^{m(t-t_0)}$$

$$x'_3(t, \beta) = \left(\frac{a'_2(\beta) + a'_3(\beta)}{2} \right) e^{-m(t-t_0)} - \left(\frac{a'_2(\beta) - a'_3(\beta)}{2} \right) e^{m(t-t_0)}$$

$$x'_4(t, \beta) = \left(\frac{a'_1(\beta) + a'_4(\beta)}{2} \right) e^{-m(t-t_0)} - \left(\frac{a'_1(\beta) - a'_4(\beta)}{2} \right) e^{m(t-t_0)}$$

IV. EXAMPLE

Example 4.1 Consider the fuzzy ordinary differential equation $\frac{dx}{dt} = x$, $x(0) = (3, 5, 8, 11, 13; 1, 5, 8, 11, 15)$

Solution:

$$x_1(t, \alpha) = (4\alpha + 3)e^t, \quad x_4(t, \alpha) = (-4\alpha + 13)e^t, \quad \alpha \in [0, 0.5]$$

$$x_2(t, \alpha) = (6\alpha + 2)e^t, \quad x_3(t, \alpha) = (-6\alpha + 14)e^t, \quad \alpha \in [0.5, 1]$$

$$x'_1(t, \beta) = (-8\beta + 9)e^t, \quad x'_4(t, \beta) = (8\beta + 7)e^t, \quad \beta \in [0.5, 1]$$

$$x'_2(t, \beta) = (-6\beta + 8)e^t, \quad x'_3(t, \beta) = (6\beta + 8)e^t, \quad \beta \in [0, 0.5]$$

Here

$$\frac{dx_1(t, \alpha)}{d\alpha} > 0, \frac{dx_4(t, \alpha)}{d\alpha} < 0, \frac{dx_2(t, \alpha)}{d\alpha} > 0, \frac{dx_3(t, \alpha)}{d\alpha} < 0, \frac{dx'_1(t, \beta)}{d\beta} < 0, \frac{dx'_4(t, \beta)}{d\beta} > 0, \frac{dx'_2(t, \beta)}{d\beta} < 0, \frac{dx'_3(t, \beta)}{d\beta} > 0$$

Hence the solution is a strong solution.

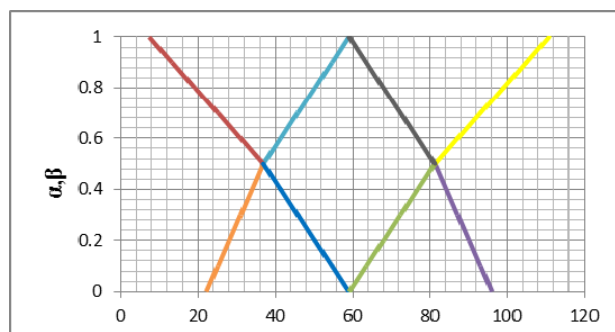


Figure 1. Graphical representation of the solution for t=2

The solution obtained is PIFN which is given by $\tilde{A}^i = (e^t, 3e^t, 4e^t, 8e^t, 11e^t; 0.5e^t, 3, 4e^t, 8e^t, 12e^t)$

The membership and non-membership function are as follows

$$\mu_{\tilde{A}_p^i}(x) = \begin{cases} \frac{x-3e^t}{4e^t}, & 3e^t \leq x \leq 5e^t \\ \frac{x-2e^t}{6e^t}, & 5e^t \leq x \leq 8e^t \\ \frac{14e^t-x}{6e^t}, & 8e^t \leq x \leq 11e^t \\ \frac{13e^t-x}{4e^t}, & 11e^t \leq x < 13e^t \\ 0, & x < a_1, a_5 \leq x \end{cases} \quad \mathcal{G}_{\tilde{A}_p^i}(x) = \begin{cases} \frac{9e^t-x_1}{8e^t}, & e^t \leq x \leq 5e^t \\ \frac{8e^t-x}{6e^t}, & 5e^t \leq x \leq 8e^t \\ \frac{x-8e^t}{6e^t}, & 8e^t \leq x \leq 11e^t \\ \frac{x-7e^t}{8e^t}, & 11e^t \leq x \leq 15e^t \\ 1, & x < b_1, b_5 < x \end{cases}$$

Example 4.2 Consider the fuzzy ordinary differential equation $\frac{dx}{dt} = x$ with the initial

condition $\tilde{x}(0) = (2, 4, 7, 11, 16; 0.5, 4, 7, 11, 17)$

Solution: The solution is given by

$$x_1(t, \alpha) = (-3\alpha + 9)e^{-3t} + (7\alpha - 7)e^{3t}$$

$$x_2(t, \alpha) = (-\alpha + 8)e^{-3t} + (7\alpha - 7)e^{3t}$$

$$x_3(t, \alpha) = (-\alpha + 8)e^{-3t} - (7\alpha - 7)e^{3t}$$

$$x_4(t, \alpha) = (-3\alpha + 9)e^{-3t} - (7\alpha - 7)e^{3t}$$

$$x'_1(t, \beta) = (2.5\beta + 6.25)e^{-3t} + (-9.5\beta + 1.25)e^{3t}$$

$$x'_2(t, \beta) = (\beta + 7)e^{-3t} + (-7\beta)e^{3t}$$

$$x'_3(t, \beta) = (\beta + 7)e^{-3t} - (-7\beta)e^{3t}$$

$$x'_4(t, \beta) = (2.5\beta + 6.25)e^{-3t} - (-9.5\beta + 1.25)e^{3t}$$

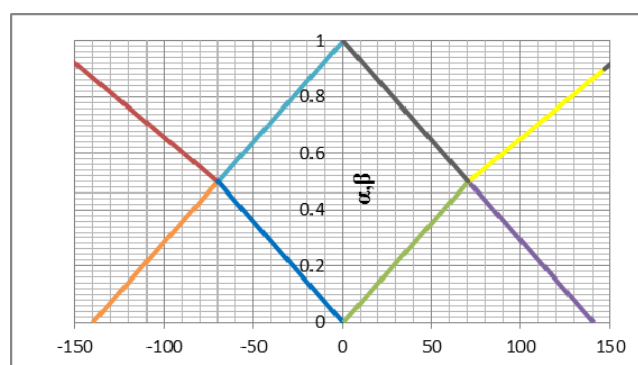


Figure 2. Graphical representation of the solution for $t=3$

The solution is a pentagonal intuitionistic fuzzy number which can be expressed as

$$\tilde{A}_p^i = (9e^{-3t} - 7e^{3t}, 7.5e^{-3t} - 3.5e^{3t}, 7e^{3t}, 7.5e^{-3t} + 3.5e^{3t}, 9e^{-3t} + 7e^{3t}; 8.75e^{-3t} - 8.25e^{3t}, 7.5e^{-3t} - 3.5e^{3t}, 7.5e^{-3t}, 7.5e^{-3t} + 3.5e^{3t}, 8.75e^{-3t} + 8.25e^{3t})$$

The membership function and non –membership function are follows

$$\mu_{\tilde{A}_p^i}(x) = \begin{cases} \frac{x - (9e^{-3t} - 7e^{3t})}{-3e^{-3t} + 7e^{3t}}, & 9e^{-3t} - 7e^{3t} \leq x \leq 7.5e^{-3t} - 3.5e^{3t} \\ \frac{x - (8e^{-3t} - 7e^{3t})}{-e^{-3t} + 7e^{3t}}, & 7.5e^{-3t} - 3.5e^{3t} \leq x \leq 7.5e^{-3t} \\ \frac{(8e^{-3t} - 7e^{3t}) - x}{e^{-3t} + 7e^{3t}}, & 7.5e^{-3t} \leq x \leq 7.5e^{-3t} + 3.5e^{3t} \\ \frac{(9e^{-3t} + 7e^{3t}) - x}{3e^{-3t} + 7e^{3t}}, & 7.5e^{-3t} + 3.5e^{3t} \leq x < 9e^{-3t} + 7e^{3t} \\ 0, & x < 9e^{-3t} - 7e^{3t}, 9e^{-3t} + 7e^{3t} \leq x \end{cases}$$

$$g_{\tilde{A}_p^i}(x) = \begin{cases} \frac{x - (6.2e^{-3t} + 1.25e^{3t})}{2.5e^{-3t} - 9.5e^{3t}}, & 8.75e^{-3t} - 8.25e^{3t} \leq x \leq 7.5e^{-3t} - 3.5e^{3t} \\ \frac{x - 7e^{-3t}}{e^{-3t} - 7e^{3t}}, & 7.5e^{-3t} - 3.5e^{3t} \leq x \leq 7e^{-3t} \\ \frac{x - 7e^{-3t}}{e^{-3t} + 7e^{3t}}, & 7e^{-3t} \leq x \leq 7.5e^{-3t} + 3.5e^{3t} \\ \frac{x - (6.2e^{-3t} - 1.25e^{3t})}{2.5e^{-3t} + 9.5e^{3t}}, & 7.5e^{-3t} + 3.5e^{3t} \leq x \leq 8.75e^{-3t} + 8.25e^{3t} \\ 1, & x < 8.75e^{-3t} - 8.25e^{3t}, 8.75e^{-3t} + 8.25e^{3t} < x \end{cases}$$

V. APPLICATIONS

Application 5.1 Sales model

The rate of increase in sales of mobile phones y (in millions) of a particular company per year was assumed to be proportional to y itself, then what will be the sales of mobile phone of that company after 10 years if the initial sales are (1020,1030,1050,1080,1120;1015,1030,1050,1080,1130) considering the proportionality as 0.05

Solution: $\frac{dy}{dt} = ky, k = 0.05, y(0) = (1020,1030,1050,1080,1120;1015,1030,1050,1080,1130)$

$$y_1(t, \alpha) = (20\alpha + 1020)e^{0.05t}, \quad y_4(t, \alpha) = (-80\alpha + 1120)e^{0.05t}, \quad \alpha \in [0, 0.5]$$

$$y_2(t, \alpha) = (40\alpha + 1010)e^{0.05t}, \quad y_3(t, \alpha) = (-60\alpha + 1110)e^{0.05t}, \quad \alpha \in [0.5, 1]$$

$$y_1'(t, \beta) = (-30\beta + 1045)e^{0.05t}, \quad y_4'(t, \beta) = (100\beta + 1030)e^{0.05t}, \quad \beta \in [0.5, 1]$$

$$y_2'(t, \beta) = (-40\beta + 1050)e^{0.05t}, \quad y_3'(t, \beta) = (60\beta + 1050)e^{0.05t}, \quad \beta \in [0, 0.5]$$

The solution is a pentagonal intuitionistic fuzzy number which is expresses as

$$\tilde{A}^i = (1020e^{0.05t}, 1030e^{0.05t}, 1050e^{0.05t}, 1080e^{0.05t}, 1120e^{0.05t}, 1015e^{0.05t}, 1030e^{0.05t}, 1050e^{0.05t}, 1080e^{0.05t}, 1130e^{0.05t})$$

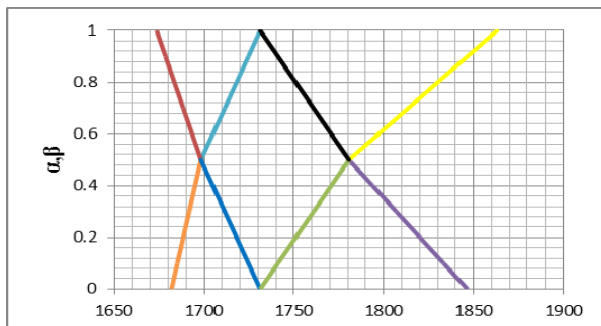


Figure 3. Graphical representation of the solution at $t=10$

The membership function and non –membership function are follows

$$\mu_{\tilde{A}_t^i}(x) = \begin{cases} \frac{x-1020e^{0.05t}}{20e^{0.05t}}, & 1020e^{0.05t} \leq x \leq 1030e^{0.05t} \\ \frac{x-1010e^{0.05t}}{40e^{0.05t}}, & 1030e^{0.05t} \leq x \leq 1050e^{0.05t} \\ \frac{1010e^{0.05t}-x}{60e^{0.05t}}, & 1050e^{0.05t} \leq x \leq 1080e^{0.05t} \\ \frac{1120e^{0.05t}-x}{80e^{0.05t}}, & 1080e^{0.05t} \leq x < 1120e^{0.05t} \\ 0, & x < 1020e^{0.05t}, 1120e^{0.05t} \leq x \end{cases} \quad \vartheta_{\tilde{A}_t^i}(x) = \begin{cases} \frac{1045e^{0.05t}-x}{30e^{0.05t}}, & 1015e^{0.05t} \leq x \leq 1030e^{0.05t} \\ \frac{1050e^{0.05t}-x}{40e^{0.05t}}, & 1030e^{0.05t} \leq x \leq 1050e^{0.05t} \\ \frac{x-1050e^{0.05t}}{60e^{0.05t}}, & 1050e^{0.05t} \leq x \leq 1080e^{0.05t} \\ \frac{x-1030e^{0.05t}}{100e^{0.05t}}, & 1080e^{0.05t} \leq x < 1130e^{0.05t} \\ 1, & x < 1015e^{0.05t}, 1130e^{0.05t} < x \end{cases}$$

Application 5.2 Illiteracy rate model: The number of illiteracy (x) in of a particular city is gradually reduces yearly, the rate of change of illiteracy with respect to year is assumed to be proportional to the x itself. If initially the number of illiteracy(in millions) is (50,60,75,95,125;55,60,75,95,130).what will be x after some time 20 years ?(considering the constant of proportionality $k=0.01$)

Solution: $\frac{dx}{dt} = kx, k = 0.01, \tilde{x}(0) = (50, 60, 75, 95, 125; 55, 60, 75, 95, 130)$

$$x_1(t, \alpha) = (-20\alpha + 87.5)e^{-0.01t} + (40\alpha - 37.5)e^{0.01t}, \alpha \in [0, 0.5]$$

$$x_2(t, \alpha) = (-5\alpha + 80)e^{-0.01t} + (35\alpha - 35)e^{0.01t}, \alpha \in [0.5, 1]$$

$$x_3(t, \alpha) = (-5\alpha + 80)e^{-0.01t} - (35\alpha - 35)e^{0.01t}, \alpha \in [0.5, 1]$$

$$x_4(t, \alpha) = (-20\alpha + 87.5)e^{-0.01t} - (40\alpha - 37.5)e^{0.01t}, \alpha \in [0, 0.5]$$

$$x'_1(t, \beta) = (30\beta + 62.5)e^{-0.01t} + (-40\beta + 2.5)e^{0.01t}, \beta \in [0.5, 1]$$

$$x'_2(t, \beta) = (5\beta + 75)e^{-0.01t} - 35\beta e^{0.01t}, \beta \in [0, 0.5]$$

$$x'_3(t, \beta) = (5\beta + 75)e^{-0.01t} + (35\beta)e^{0.01t}, \beta \in [0, 0.5]$$

$$x'_i(t, \beta) = (30\beta + 62.5)e^{-0.01t} - (-40\beta + 2.5)e^{0.01t}, \beta \in [0.5, 1]$$

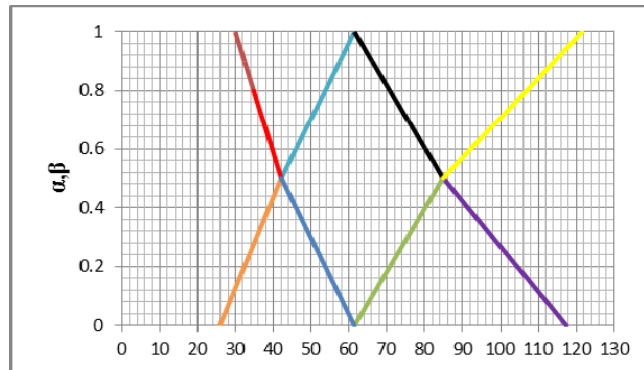


Figure 4. Graphical representation of the solution at t= 20 years.

The solution is a pentagonal intuitionistic fuzzy number which can be expressed as

$$\tilde{A}_p^i = (87.5e^{-0.01t} - 37.5e^{0.01t}, 77.5e^{-0.01t} - 17.5e^{0.01t}, 75e^{-0.01t}, 77.5e^{-0.01t} + 17.5e^{0.01t}, 87.5e^{-0.01t} + 37.5e^{0.01t}; 92.5e^{-0.01t} - 37.5e^{0.01t}, 77.5e^{-0.01t} - 17.5e^{0.01t}, 75e^{-0.01t}, 77.5e^{-0.01t} + 17.5e^{0.01t}, 92.5e^{-0.01t} + 37.5e^{0.01t})$$

The membership function and non –membership function are follows

$$\mu_{\tilde{A}_p^i}(x) = \begin{cases} \frac{x - (87.5e^{-0.01t} - 37.5e^{0.01t})}{-20e^{-0.01t} + 40e^{0.01t}}, & 87.5e^{-0.01t} - 37.5e^{0.01t} \leq x \leq 77.5e^{-0.01t} - 17.5e^{0.01t} \\ \frac{x - (80e^{-0.01t} + 35e^{0.01t})}{-5e^{-0.01t} + 35e^{0.01t}}, & 77.5e^{-0.01t} - 17.5e^{0.01t} \leq x \leq 75e^{-0.01t} \\ \frac{(80e^{-0.01t} + 35e^{0.01t}) - x}{5e^{-0.01t} + 35e^{0.01t}}, & 75e^{-0.01t} \leq x \leq 77.5e^{-0.01t} + 17.5e^{0.01t} \\ \frac{(87.5e^{-0.01t} - 37.5e^{0.01t}) - x}{20e^{-0.01t} + 40e^{0.01t}}, & 77.5e^{-0.01t} + 17.5e^{0.01t} \leq x < 87.5e^{-0.01t} + 37.5e^{0.01t} \\ 0, & x < 87.5e^{-0.01t} - 37.5e^{0.01t}, 87.5e^{-0.01t} + 37.5e^{0.01t} \leq x \end{cases}$$

$$\vartheta_{\tilde{A}_p^i}(x) = \begin{cases} \frac{x - (62.5e^{-0.01t} + 2.5e^{0.01t})}{30e^{-0.01t} - 40e^{0.01t}}, & 92.5e^{-0.01t} - 37.5e^{0.01t} \leq x \leq 77.5e^{-0.01t} - 17.5e^{0.01t} \\ \frac{x - 75e^{-0.01t}}{5e^{-0.01t} - 35e^{0.01t}}, & 77.5e^{-0.01t} - 17.5e^{0.01t} \leq x \leq 75e^{-0.01t} \\ \frac{x - 75e^{-0.01t}}{5e^{-0.01t} + 35e^{0.01t}}, & 75e^{-0.01t} \leq x \leq 77.5e^{-0.01t} + 17.5e^{0.01t} \\ \frac{x - (62.5e^{-0.01t} - 2.5e^{0.01t})}{30e^{-0.01t} + 40e^{0.01t}}, & 77.5e^{-0.01t} + 17.5e^{0.01t} \leq x \leq 92.5e^{-0.01t} + 37.5e^{0.01t} \\ 1, & x < 92.5e^{-0.01t} - 37.5e^{0.01t}, 92.5e^{-0.01t} + 37.5e^{0.01t} < x \end{cases}$$

VI. CONCLUSION

In this paper, we have arrived solution to a first order homogeneous differential equation whose initial condition as pentagonal intuitionistic fuzzy number .Two example problems are solved using intuitionistic fuzzy concepts the graph of the solution is plotted .The obtained graph is in the form of intuitionistic pentagonal type which validates our results. This work can be applied to higher order differential equation and for non- homogeneous differential equation.

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