Determinantal Identities of Generalized Fibonacci-Like Sequence

Yogesh Kumar Gupta
School of Studies in Mathematics, Vikram University, Ujjain-456010 (M. P.), India
yogeshgupta.880@rediffmail.com

Omprakash Sikhwal
Devanshi Tutorial, Keshw Kunj, Mandsaur (M.P.), India
opbhsikhwal@rediffmail.com

Kiran Sisodiya
School of Studies in Mathematics, Vikram University, Ujjain (M. P.), India
sisodiya.kiran4@gmail.com

Abstract- Determinants have played a significant part in various areas in mathematics. For instance, they are quite useful in the analysis and solution of system of linear equations. There are different perspectives on the study of determinant. In this paper we present some determinant identities of generalized Fibonacci-Like sequence are presented.

Keywords: Fibonacci sequence, Lucas sequence, Fibonacci-Like sequence, generalized Fibonacci-Like sequence.

I. INTRODUCTION

It is well known that the Fibonacci numbers and polynomials are of great importance in the study of many subjects such as algebra, geometry, combinatorics, approximation theory, graph theory and number theory itself. They occur in a variety of other fields such as finance, art, architecture, music, etc.

There is a long tradition of using matrices and determinants to study Fibonacci numbers. Problems on determinants of Fibonacci sequence and Lucas sequence. Cahill and Narayan [3] show how Fibonacci and Lucas numbers arise as determinants of some tridiagonal matrices. Bicknell –Johnson and Spears [2] use elementary matrix operations and determinants to generate classes of identities for generalized Fibonacci numbers. T. Koshy [8] explained two chapters on the use of matrices and determinants in Fibonacci numbers.

One may notice several practical and effective instruments for calculating determinants in the nice survey articles [12] and [5]. Much attention has been paid to the evaluation of determinants of matrices, especially when their entries are given recursively [11].

Macfarlane [11] use the property for determinants to give new identities involving Fibonacci and related number is defined by the recurrence relation,

Fibonacci sequence is defined as

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \quad with\ F_0 = 0, \quad F_1 = 1$$

Where F_n is a n^{th} number of sequence.
Lucas sequence is defined as

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2 \quad with\ L_0 = 2, \quad L_1 = 1$$

Where L_n is a n^{th} number of sequence.
The Fibonacci-Like sequence is defined by recurrence relation,

$$S_n = S_{n-1} + S_{n-2}, \quad n \geq 2 \quad with\ S_0 = 2, \quad S_1 = 2$$

first few numbers of Fibonacci-Like sequences are 2, 2, 4, 6, 10, 16, and so on.

In this section some determinant identities of generalized Fibonacci-Like sequence are presented. Generalized Fibonacci-Like sequence is introduced and defined by the recurrence relation

$$M_n = M_{n-1} + M_{n-2} \quad n \geq 2 \text{ with } M_0 = 2, M_1 = s+1,$$

where s being a fixed integers.

The first few terms are as follows:

$M_0 = 2,$
$M_1 = s+1,$
$M_2 = s+3,$
$M_3 = 2s+4,$
$M_4 = 3s+7,$
$M_5 = 5s+11,$
$M_6 = 8s+18,$
$B_7 = 13s+29$ *and so on.*

II.

III. DETERMINANTAL IDENTITIES

In this section, we derive some determinant identities of generalized Fibonacci-Like sequence are presented.

Theorem (2.1). For any integers $n \geq 0$, prove that

$$\begin{vmatrix} M_{n+1} & M_{n+2} & M_{n+3} \\ M_{n+4} & M_{n+5} & M_{n+6} \\ M_{n+7} & M_{n+8} & M_{n+9} \end{vmatrix} = 0.$$

Proof.

$$\text{Let } \Delta = \begin{vmatrix} M_{n+1} & M_{n+2} & M_{n+3} \\ M_{n+4} & M_{n+5} & M_{n+6} \\ M_{n+7} & M_{n+8} & M_{n+9} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we get

$$\Delta = \begin{vmatrix} M_{n+3} & M_{n+2} & M_{n+3} \\ M_{n+6} & M_{n+5} & M_{n+6} \\ M_{n+9} & M_{n+8} & M_{n+9} \end{vmatrix}$$

Since two columns are identical, thus we obtained required result.

Theorem (2.2). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n - M_{n+1} & M_{n+1} - M_{n+2} & M_{n+2} - M_n \\ M_{n+1} - M_{n+2} & M_{n+2} - M_n & M_n - M_{n+1} \\ M_{n+2} - M_n & M_n - M_{n+1} & M_{n+1} - M_{n+2} \end{vmatrix} = 0.$$

Proof. $$\text{Let } \Delta = \begin{vmatrix} M_n - M_{n+1} & M_{n+1} - M_{n+2} & M_{n+2} - M_n \\ M_{n+1} - M_{n+2} & M_{n+2} - M_n & M_n - M_{n+1} \\ M_{n+2} - M_n & M_n - M_{n+1} & M_{n+1} - M_{n+2} \end{vmatrix}.$$

By applying $C_1 \to C_1 + C_2 + C_3$ and expanding along first row, we obtained required result.

Theorem (2.3). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ M_n & M_{n+1} & M_{n+2} \\ M_{n+1} + M_{n+2} & M_n + M_{n+2} & M_n + M_{n+1} \end{vmatrix} = 0.$$

Proof. $$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ M_n & M_{n+1} & M_{n+2} \\ M_{n+1} + M_{n+2} & M_n + M_{n+2} & M_n + M_{n+1} \end{vmatrix}.$$

Applying $R_3 \to R_3 + R_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ M_n & M_{n+1} & M_{n+2} \\ 2M_{n+2} & 2M_{n+2} & 2M_{n+2} \end{vmatrix}.$$

Taking common out $2M_{n+2}$ from third row,

$$\Delta = 2M_{n+2} \begin{vmatrix} 1 & 1 & 1 \\ M_n & M_{n+1} & M_{n+2} \\ 1 & 1 & 1 \end{vmatrix}.$$

Since two rows are identical, thus we obtained required result.

Theorem (2.4). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n & M_n + M_{n+1} & M_n + M_{n+1} + M_{n+2} \\ 2M_n & 2M_n + 3M_{n+1} & 2M_n + 3M_{n+1} + 4M_{n+2} \\ 3M_n & 3M_n + 6M_{n+1} & 3M_n + 6M_{n+1} + 12M_{n+2} \end{vmatrix} = 3M_n M_{n+1} M_{n+2}.$$

Proof. Let $\Delta = \begin{vmatrix} M_n & M_n + M_{n+1} & M_n + M_{n+1} + M_{n+2} \\ 2M_n & 2M_n + 3M_{n+1} & 2M_n + 3M_{n+1} + 4M_{n+2} \\ 3M_n & 3M_n + 6M_{n+1} & 3M_n + 6M_{n+1} + 12M_{n+2} \end{vmatrix}$.

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get

$$\Delta = \begin{vmatrix} M_n & M_n + M_{n+1} & M_n + M_{n+1} + M_{n+2} \\ 0 & M_{n+1} & M_{n+1} + 2M_{n+2} \\ 0 & 3M_{n+1} & 3M_{n+1} + 9M_{n+2} \end{vmatrix}.$$

Applying $R_3 \to R_3 - 3R_2$ and expanding along first row, we obtained required result.

Theorem (2.5). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} 0 & M_n M_{n+1}^2 & M_n M_{n+2}^2 \\ M_n^2 M_{n+1} & 0 & M_{n+1} M_{n+2}^2 \\ M_n^2 M_{n+2} & M_{n+2} M_{n+1}^2 & 0 \end{vmatrix} = 2M_n^3 M_{n+1}^3 M_{n+2}^3.$$

Proof. Let $\Delta = \begin{vmatrix} 0 & M_n M_{n+1}^2 & M_n M_{n+2}^2 \\ M_n^2 M_{n+1} & 0 & M_{n+1} M_{n+2}^2 \\ M_n^2 M_{n+2} & M_{n+2} M_{n+1}^2 & 0 \end{vmatrix}$.

Taking common out $M_n^2, M_{n+1}^2, M_{n+2}^2$ from C_1, C_2, C_3 respectively, we get

$$\Delta = M_n^2 M_{n+1}^2 M_{n+2}^2 \begin{vmatrix} 0 & M_n & M_n \\ M_{n+1} & 0 & M_{n+1} \\ M_{n+2} & M_{n+2} & 0 \end{vmatrix}.$$

Taking common out M_n, M_{n+1}, M_{n+2} from R_1, R_2, R_3 respectively and expanding along first row, we obtained required result.

Theorem (2.6). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n & F_n & 1 \\ M_{n+1} & F_{n+1} & 1 \\ M_{n+2} & F_{n+2} & 1 \end{vmatrix} = \left[F_n M_{n+1} - M_n F_{n+1}\right].$$

Proof. Let $\Delta = \begin{vmatrix} M_n & F_n & 1 \\ M_{n+1} & F_{n+1} & 1 \\ M_{n+2} & F_{n+2} & 1 \end{vmatrix}$

Assume $M_n = a,\ M_{n+1} = b,\ F_n = p,\ F_{n+1} = q\ then\ M_{n+2} = a + b\ and\ F_{n+2} = p + q$

Now substituting the above values in determinant, we get

$$\Delta = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ -a & -p & 0 \\ a+b & p+q & 1 \end{vmatrix} = (pb - aq).$$

Substituting the values of a, b, p and q, we get required result.

Similarly following identities can be derived:

Theorem (2.7). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n & M_{n+1} & M_{n+2} \\ M_{n+2} & M_n & M_{n+1} \\ M_{n+1} & M_{n+2} & M_n \end{vmatrix} = 2(M_n^3 + M_{n+1}^3).$$

Theorem (2.8). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n & L_n & 1 \\ M_{n+1} & L_{n+1} & 1 \\ M_{n+2} & L_{n+2} & 1 \end{vmatrix} = 2(L_n M_{n+1} - M_n L_{n+1}).$$

Theorem (2.9). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n + M_{n+1} & M_{n+1} + M_{n+2} & M_{n+2} + M_n \\ M_{n+2} & M_n & M_{n+1} \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Theorem (2.10). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} 1+M_n & M_{n+1} & M_{n+2} \\ M_n & 1+M_{n+1} & M_{n+2} \\ M_n & M_{n+1} & 1+M_{n+2} \end{vmatrix} = 1 + M_n + M_{n+1} + M_{n+2}.$$

IV. CONCLUSIONS

In this paper, some determinant identities of generalized Fibonacci-Like sequence are presented. Entries of determinants are satisfying the recurrence relation of generalized Fibonacci-Like sequence also some determinant identities have been established and derived.

ACKNOWLEDGMENT

Guidance during preparation of this research paper fills me with a sense of gratitude. Firstly, I express my gratitude to my co supervisor Dr. Mamta Singh for their generous support.

REFERENCES.

[1] Benjamin A., Cameron N. and Quinn J., Fibonacci Determinants-A Combinatorial Approach, Fibonacci Quarterly, Vol. 45, No.1 (2007), 39-55.
[2] Bicknell-Johnson M. and Spears C. P., Classes of Identities for the Generalized Fibonacci number Gn= Gn-1 + Gn-2, n ≥ 2. from Matrices with Constant valued Determinants, Fibonacci Quarterly, Vol. 34, No.2 (1996), 121-128.
[3] Cahill N. and Narayan D., Fibonacci and Lucas numbers s Tridigonal Matrix Determinants, Fibonacci Quarterly, Vol. 42, No.3 (2004), 216-221.
[4] Gupta, Y.K, Sikhwal, O. and Singh, M., Determinantal Identities of Fibonacci, Lucas and Generalized Fibonacci-Lucas Sequence, MAYFEB Journal of Mathematics, Vol. 2 (2016) - Pages 17-23.
[5] Gupta, Y.K, Singh, M., and Sikhwal, O. , Generalized Fibonacci – Like Sequence Associated with Fibonacci and Lucas Sequences, Turkish Journal of Analysis and Number Theory, Vol. 2, No. 6 (2014), 233-238.
[6] Gupta, Y.K, Badshah, V. H. , Singh, M., and Sisodiya, K. , Generalized Additive Coupled Fibonacci Sequences of Third order and Some Identities'', International Journal of Innovative Science, Engineering & Technology, Vol. 2 Issue 3, March 2015, 80-55.
[7] Horadam, A. F., A Generalized Fibonacci sequence, American Mathematical Monthly, Vol. 68. (5), 1961, 455-459.
[8] Koshy, T., Fibonacci and Lucas Numbers with Applications, Wiley-Interscience Publication, New York (2001).
[9] Krattenthaler C., Advanced determinant calculus, Seminaire Lotharingien Combin, Article, b42q, 67, 1999.
[10] Krattenthaler C., Advanced determinant calculus: A Complement, Liner Algebra Appl., 68- 166.
[11] Macfarlane A. J., Use of Determinants to Present Identities Involving Fibonacci and Related Numbers, Fibonacci Quarterly, Vol. 48, No.1(2010), 68-76.
[12] Singh, B., Sikhwal, O. and Gupta, Y.K., Generalized Fibonacci-Lucas Sequence, Turkish Journal of Analysis and Number Theory, Vol.2, No.6 (2014), 193-197.
[13] Singh, M., Gupta, Y.K., and Sisodiya, K., Generalization of Fibonacci Sequence and Related Properties, "Research Journal of Computation and Mathematics, Vol. 3, No. 2, (2015), 12-18.
[14] Singh, M., Gupta, Y., and Sikhwal, O., Generalized Fibonacci- Lucas Sequence Its Properties, Global Journal of Mathematical Analysis, Germany, Vol. 2, No. 3. (2014), 160-168.
[15] Singh, M., Sikhwal, O. and Gupta, Y., Identities of Generalized Fibonacci-Like Sequence, Turkish Journal of Analysis and Number Theory, Vol. 2, No. 5 (2014), 170-175.
[16] Spivey M. Z., Fibonacci Identities via the Determinant sum property, College Mathematics Journal, Vol.37, No.4 (2006), 286-289.