A Very Flexible Weibull Extended Distribution

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Abstract- **The present study considers a new function to introduce a very flexible lifetime distribution. The new distribution is introduced by considering a linear scheme of the two logarithms of cumulative hazard functions and is named as very flexible Weibull extended distribution. The new model is very flexible and accommodates increasing, unimodal, modified unimodal or bathtub shapes failure rate. Different structural properties along with the estimation using maximum likelihood method will be discussed. In order to illustrate the usefulness of the proposed model, an application to a well-known lifetime data set will be analyzed, and the goodness of fit result of the proposed model will be compared with Weibull and eleven other well-known extensions of the Weibull model.**

Keywords: Reducing the number of parameters; Bathtub shape; Moment generating function; Order statistics; Maximum likelihood estimation

I. INTRODUCTION

In the exercise of analyzing real phenomena one may frequently uses the Exponential, Rayleigh, or Weibull distributions. These distributions offers several desirable statistical properties and are frequently used to model lifetime data. Between these lifetime distributions, the Weibull model was originally introduced by Weibull [23], a Swedish physicist, and he utilized it to represent the distribution of breaking strength of materials. The probability density function (PDF) of the two parameters Weibull distribution is given by

$$g(z) = \beta\theta z^{\beta-1} e^{-\theta z^{\beta}}, \qquad z,\theta,\beta>0. \tag{1}$$

The hazard function (HF) of the Weibull model is

$$h(z) = \beta\theta z^{\beta-1}. \tag{2}$$

The Weibull distribution is the most popular one and offers the characteristics of other aging distributions such as Rayleigh and exponential distributions. Instead of using Rayleigh and Exponential distribution, the Weibull distribution is a very suitable model to use for modeling lifetime data, where the failure rate of the data behaves monotonically. However, with a limited failure rate functions, the Weibull model is inappropriate to use for modeling real phenomena exhibiting non-monotonic hazard shapes such as, unimodal, modified unimodal or bathtub shaped hazard rates. Between the non-monotonic hazard function, the bathtub shaped hazard function is very common and having numerous applications in the literature. For example, in bio-medical analysis: the human mortality rate is observed to have bathtub shaped hazard function. Whereas, in reliability engineering: the lifecycle of electronic components is detected to have bathtub shaped failure rate function. Due to practical utility in bio-medical and reliability disciplines, a number of researchers have been working to propose to new extensions of the Weibull model with number of parameters extending from 2 to 5 to obtain non-monotonic failure rate function. For example, the two-parameter flexible Weibull extension (FWEx) of Bebbington et al. [7] capable of modeling with increasing, decreasing or bathtub shaped. A three parameter model, named as exponentiated Weibull (EW) distribution, was studied by Mudholkar and Srivastava [15]. Zhang and Xie [26] studied the truncated Weibull distribution having bathtub shaped hazard function. Another three-parameter lifetime model studied by Marshall and Olkin [14]. A five parameter model, called new modified Weibull (NMW) distribution of Almalki and Yuan [6], Xie et al. [25] proposed a three parameter model named as modified Weibull extension (MWEx) with a bathtub shaped failure rate function. A four-parameter model, the additive Weibull (AW) distribution of Xie and Lai [24], a three parameter modified Weibull (MW) distribution by Sarhan and Zaindin [17], a four-parameter beta Weibull (BW) distribution of Famoye et al. [10]. Another four parameter lifetime model, Kumaraswamy Weibull distribution (Ku-W) studied by Cordeiro et al. [8], Five-parameter modified Weibull (MW) distributions by Phani's modified Weibull [16], and beta modified Weibull (BMW) proposed by Silva et al. [19], The latest examples including the beta generalized Weibull (BGW) distribution of Singla et al. [20], the generalized Gomprtz (GG) distribution due to El-Gohary et al. [9], new flexible Weibull (NFW) and flexible Weibull (FW) distributions of Ahmad and Hussain [2] and [3], and generalized flexible Weibull extension (GFWEx) distribution of Ahmad and Iqbal [4].

It is a very useful method to mix two different survival functions (one with increasing and other with decreasing hazards), and produce a new function as

$$S(z)=\xi S_1(z)+(1-\xi)S_2(z),$$

where $0 < \xi < 1$, it is better known as a mixture of models/distributions, or

$$S(z)=\xi S_1(z)+\gamma S_2(z), \tag{3}$$

with parameters $\xi,\gamma>0$.

In term of cumulative HF, the cumulative distribution function (CDF) can be written as

$$G(z)=1-e^{-H(z)}, \tag{4}$$

where the cumulative HF denoted by $H(z)$ fulfils the following settings as

i. $H(z)$ is nonnegative as well as increasing function ,

ii. $\lim_{z \to 0} H(z) \to 0$ and $\lim_{z \to \infty} H(z) \to \infty$.

Hence, we can also mix two cumulative hazard functions in order to generate a new function as

$$H(z) = \theta H_1(z) + \gamma H_2(z), \tag{5}$$

Here in (5), the $H(z)$ is bounded. However, the present study try to relax the boundary conditions, therefore, the function $logH(z)$ is used, instead of $H(z)$ to propose very flexible function. Because it would be more appropriate to use $logH(z)$ rather than $H(z)$ in order to generate a very flexible model. Hence, one can write the expression in (5) in the form given by

$$H(z) = H_1^{\theta}(z) \times H_2^{\gamma}(z), \tag{6}$$

after using $logH(z)$ instead of $H(z)$ in (6), one may get

$$logH(z) = \theta logH_1(z) + \gamma logH_2(z). \tag{7}$$

Here, in (7), one parameter i.e. γ is fixed as $\gamma = 1$, and propose a very flexible lifetime model with fewer parameters. The expression in (7), becomes

$$logH(z) = \theta logH_1(z) + logH_2(z). \tag{8}$$

Instead of fixing one parameter constant, the proposed model still provides greater distributional flexibility. A combination of the logarithm of the two cumulative HF's defined as z and $\left\{ -\left(\frac{1}{z}\right)^{\lambda} \right\}$ is used to generate a new very flexible lifetime model. The expression in (8), now becomes

$$H(z) = e^{\left(\theta z - \frac{1}{z^{\lambda}}\right)}. \tag{9}$$

By using (9), in (4), one may easily obtain the CDF of very flexible Weibull extended (VFWEx) distribution.

The rest of this research article is organized as follow: Section 2, offers the definition as well as visual sketching of the new model. Section 3, contains basic properties. Section 4, 5 and 6, derives the moment generating function, probability generating function and factorial moment generating function (FMGF) of the model, respectively. Section 7, covers the estimation of the model parameters. Densities of the order statistics are discussed in section 8. In section 9, an analysis of real data set is provided. Finally, in section 10, concluding remarks about the article are provided.

II. Very Flexible Weibull Extended Distribution

Let Z be the VFWEx random variable with parameters (θ,λ). Then, the CDF of Z is given by

$$G(z;\theta,\lambda)=1-e^{-e^{\left(\theta z-\frac{1}{z^{\lambda}}\right)}}, \qquad z,\theta,\lambda>0. \tag{10}$$

Here, $\theta(>0)$, denotes the scale parameters, and $\lambda(>0)$, is a shape parameter of the model.

The density function corresponding to (10) is

$$g(z;\theta,\lambda)=\left(\theta+\frac{\lambda}{z^{\lambda+1}}\right)e^{\left(\theta z-\frac{1}{z^{\lambda}}\right)}e^{-e^{\left(\theta z-\frac{1}{z^{\lambda}}\right)}}. \tag{11}$$

The SF of VFWEx distribution is

$$S(z;\theta,\lambda)=e^{-e^{\left(\theta z-\frac{1}{z^{\lambda}}\right)}}, \tag{12}$$

with HF given by

$$h(z;\theta,\lambda)=\left(\theta+\frac{\lambda}{z^{\lambda+1}}\right)e^{\left(\theta z-\frac{1}{z^{\lambda}}\right)}.$$

The figure 1 & figure 2 visually displays the HFs of the VFWEx distribution for different parameter values.

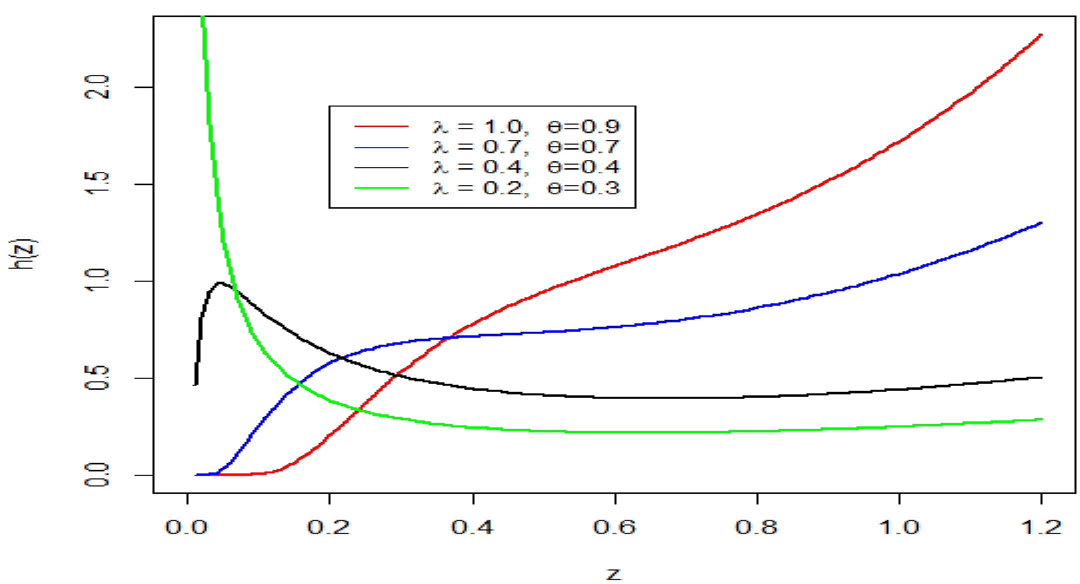

Figure 1. HF of VFWEx distribution, for some selected parameter values.

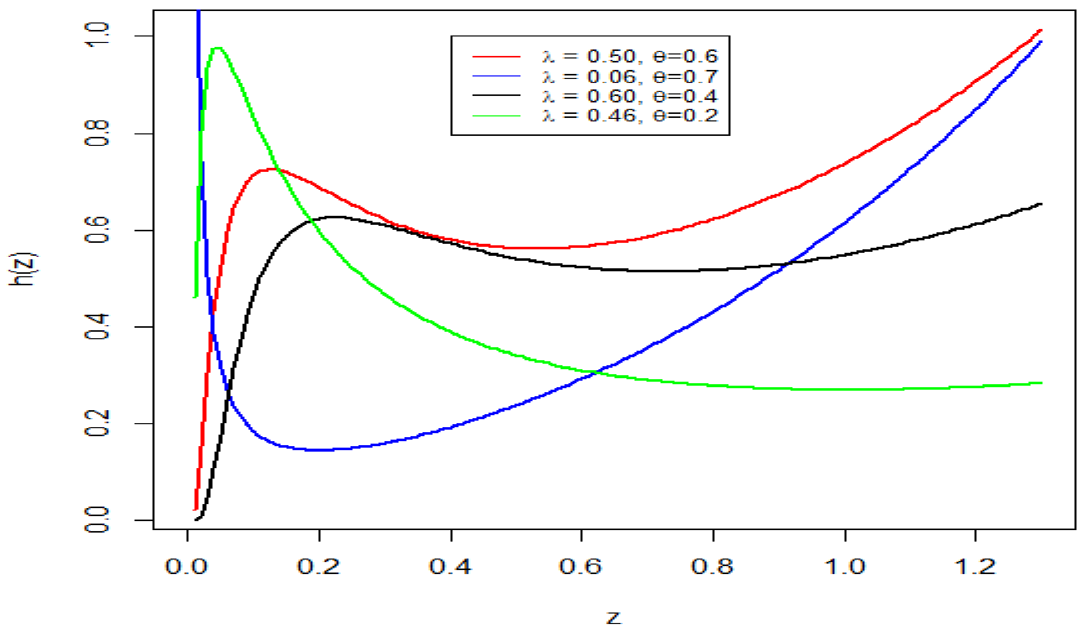

Figure 2. HF of VFWEx distribution, for some selected parameter values.

III. Basic Mathematical Properties

This section of the article provides the basic mathematical properties of VFWEx distribution.

A. *Quantile and Median*

The q^{th} quantile denoted by z_q of the VFWEx model is derived as

$$G\left(z_q\right) = q,$$

$$\left(1 - e^{-e^{\left(\theta z_q - \frac{1}{z_q^{\lambda}}\right)}}\right) = q.$$

On solving, one may get the following result,

$$-\ln\left(1-q\right) = e^{\left(\theta z_q - \frac{1}{z_q^{\lambda}}\right)},$$

$$\ln\left\{-\ln\left(1-q\right)\right\} = \left(\theta z_q - \frac{1}{z_q^{\lambda}}\right),$$

$$\left[\ln\left\{-\ln\left(1-q\right)\right\}\right]z_q^{\lambda}-\theta z_q^{\lambda+1}+1=0. \tag{13}$$

From (13), it is well clear that the solution for the q^{th} quantile of VFWEx distribution is not in closed form. Therefore, the q^{th} quantile of the new model can be obtained numerically by solving the nonlinear equation provided in (13). Using $q = 0.50$, in (13), one may easily get the median of the new distribution. While using $q = 0.25$, and $q = 0.75$, in (13), one might obtain the *1st* and *3rd* quartile of the VFWEx distribution, respectively.

B. Moments

If *Z* has the VFWEx distribution with parameters (θ,λ). Then its r^{th} moment can be derived as

$$\mu_r^{/}=\int_0^{\infty} z^r g\left(z\right)dz,$$

$$\mu_r^{/}=\int_0^{\infty} z^r \left(\theta+\frac{\lambda}{z^{\lambda+1}}\right)e^{\left(\theta z-\frac{1}{z^{\lambda}}\right)}e^{-e^{\left(\theta z-\frac{1}{z^{\lambda}}\right)}}dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\int_0^{\infty} z^r \left(\theta+\frac{\lambda}{z^{\lambda+1}}\right)\left\{e^{\left(\theta z-\frac{1}{z^{\lambda}}\right)}\right\}^{i+1}dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\int_0^{\infty} z^r \left(\theta+\frac{\lambda}{z^{\lambda+1}}\right)e^{\theta(i+1)z}\ e^{-\frac{(i+1)}{z^{\lambda}}}dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\left(i+1\right)^j}{i!\,j!}\int_0^{\infty} z^{r-j\lambda}\left(\theta+\frac{\lambda}{z^{\lambda+1}}\right)e^{\theta(i+1)z}dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\left(i+1\right)^j}{i!\,j!}\left\{\theta\int_0^{\infty} z^{r-j\lambda}e^{\theta(i+1)z}\ dz+\lambda\int_0^{\infty} z^{r-\lambda(j+1)-1}e^{\theta(i+1)z}\ dz\right\}. \tag{14}$$

Using the definition of Gamma function (Zwillinger (2014))

$$\Gamma y = z^y\int_0^{\infty} e^{tz}\ t^{y-1}dt,\ \ y,z>0.$$

Using the above gamma function definition in (14), the following result is observed.

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\left(i+1\right)^j}{i!\,j!}\left\{\theta\frac{\Gamma\left(r-j\lambda+1\right)}{\left(\theta\left(i+1\right)\right)^{r-j\lambda+1}}+\lambda\frac{\Gamma\left(r-\lambda\left(j+1\right)\right)}{\left(\theta\left(i+1\right)\right)^{r-\lambda(j+1)}}\right\}. \tag{15}$$

C. Generation of Random Numbers

The formula for generating random numbers from VFWEx distribution can be derived as

$$G(z)=R, \qquad \text{Where } R\sim U(0,1)$$

$$\left(1-e^{-e^{\left(\theta z-\frac{1}{z^{\lambda}}\right)}}\right)=R.$$

On solving, we get,

$$\left[\ln\left\{-\ln(1-R)\right\}\right]z^{\lambda}-\theta z^{\lambda+1}+1=0. \tag{16}$$

The result derived in (16) does not holds a closed form solution. Therefore, the random numbers from VFWEx distribution can be generated numerically using computer software.

IV. Moment Generating Function

Let Z follows VFWEx distribution with parameters (θ,λ). Then, the moment generating function (MGF) of Z is derived as

$$M_z(t)=\int_0^{\infty}e^{tz}g(z;\theta,\lambda)\,dz$$

$$M_z(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}z^r g(z;\theta,\lambda)\,dz$$

$$M_z(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{\prime} \tag{17}$$

By using (15), in (17), we may get the complete proof for the MGF of VFWEx distribution.

V. Probability Generating Function

Let Z follows VFWEx distribution with parameters (θ,λ). Then, the probability generating function (PGF) of Z is derived as

$$G(\gamma)=\int_0^{\infty}\gamma^z g(z;\theta,\lambda)\,dz$$

$$G(\gamma)=\sum_{r=0}^{\infty}\frac{\log^r(\gamma)}{r!}\int_0^{\infty}z^r g(z;\theta,\lambda)\,dz,$$

$$G(\gamma)=\sum_{r=0}^{\infty}\frac{\log^{r}(\gamma)}{r!}\,\mu_r^{/}. \tag{18}$$

On substituting (15), in (18), one may get the final expression for the PGF of VFWEx distribution.

VI. Factorial Moment Generating Function

The factorial moment generating function (FMGF) of VFWEx random variable can be derived as

$$H_0(1+\delta)=\int_0^{\infty}(1+\delta)^{z}\,g(z;\theta,\lambda)\,dz$$

$$H_0(1+\delta)=\sum_{r=0}^{\infty}\frac{\log^{r}(1+\delta)}{r!}\int_0^{\infty}z^{r}g(z;\theta,\lambda)\,dz,$$

$$H_0(1+\delta)=\sum_{r=0}^{\infty}\frac{\log^{r}(1+\delta)}{r!}\,\mu_r^{/}. \tag{19}$$

By substituting (15), in (19), one may have the complete solution for the FMGF of VFWEx distribution.

VII. Estimation

In this section of the article, the parameter estimation of the VFWEx model using maximum likelihood estimation (MLE) [11] procedure along with their asymptotic confidence bounds are discussed. Other possible approaches to estimate the parameters include Bayesian approach using Lindley approximation [13] or Markov chain Monte Carlo simulation (MCMC) ([18], [21]) are also available to estimate the parameters.

A. *Maximum Likelihood Estimation*

Let $Z_1, Z_2, \ldots, Z_k$ are randomly sampled from VFWEx distribution with parameters (θ,λ). Then the corresponding likelihood function is

$$L=\prod_{j=1}^{k}g(z;\theta,\lambda),$$

$$L=\prod_{j=1}^{k}\left(\theta+\frac{\lambda}{z_j^{\lambda+1}}\right)e^{\left(\theta z_j-\frac{1}{z_j^{\lambda}}\right)}e^{-e^{\left(\theta z_j-\frac{1}{z_j^{\lambda}}\right)}}.$$

The log-likelihood function is

$$\ln L = \sum_{j=1}^{k} \ln\left(\theta + \frac{\lambda}{z_j^{\lambda+1}}\right) + \sum_{j=1}^{k}\left(\theta z_j - \frac{1}{z_j^{\lambda}}\right) - \sum_{j=1}^{k} e^{\left(\theta z_j - \frac{1}{z_j^{\lambda}}\right)}. \tag{20}$$

By finding the partial derivatives of the expression given in (20) on behalf of un-known parameters, and then equating the result to zero.

$$\frac{d\ln L}{d\theta} = \sum_{j=1}^{k} \frac{z_j^{\lambda+1}}{\left(\theta z_j^{\lambda+1} + \lambda\right)} + \sum_{j=1}^{k} z_j - \sum_{j=1}^{k} z_j e^{\left(\theta z_j - \frac{1}{z_j^{\lambda}}\right)} = 0. \tag{21}$$

$$\frac{d\ln L}{d\lambda} = \sum_{j=1}^{k} \frac{\left\{z_j^{\lambda+1} - \lambda \ln\left(z_j\right) z_j^{\lambda+1}\right\}}{\left(\theta + \frac{\lambda}{z_j^{\lambda+1}}\right)\left(z_j^{\lambda+1}\right)^2} + \sum_{j=1}^{k} z_j^{-\lambda} \ln\left(z_j\right)\left\{1 - e^{\left(\theta z_j - \frac{1}{z_j^{\lambda}}\right)}\right\} = 0. \tag{22}$$

It is observed that, the expressions derived in (21) and in (22) do not have closed forms solution. Therefore, the iterating procedure newton Raphson is to be used to estimate the model parameters numerically. In the present paper, the "SANN" algorithm in R language is used to obtain numerical estimates of the model parameters.

B. Asymptotic Confidence Bounds

From (21) and (22) it is detected that these expressions are not in a closed form. So, it is relatively tough to find the exact distribution of the MLE's. Therefore, it might be pretty good to derive the asymptotic confidence limits of the unknown parameters. The most frequently and sound good approach is to assume that the MLE's $\left(\hat{\theta}, \hat{\gamma}\right)$ are approximately normally distributed having mean $\left(\theta, \lambda\right)$ and covariance matrix Σ. All of the second order derivatives for the density of VFWEx distribution exist. Thus

$$\begin{pmatrix}\hat{\theta}\\ \hat{\lambda}\end{pmatrix} \sim \mathrm{N}\left[\begin{pmatrix}\theta\\ \lambda\end{pmatrix}, \Sigma\right],$$

With

$$\Sigma = -\,\mathrm{E}\begin{bmatrix}\mathrm{V}_{\theta\theta} & \mathrm{V}_{\theta\lambda}\\ \mathrm{V}_{\lambda\theta} & \mathrm{V}_{\lambda\lambda}\end{bmatrix}^{-1},$$

Where

$$V_{\theta\theta} = \frac{\partial^2 \ln L}{\partial\theta^2}, \qquad V_{\theta\lambda} = \frac{\partial^2 \ln L}{\partial\theta\partial\lambda}$$
$$V_{\lambda\theta} = \frac{\partial^2 \ln L}{\partial\lambda\partial\theta}, \qquad V_{\lambda\lambda} = \frac{\partial^2 \ln L}{\partial\lambda^2}$$

Since Σ consist of unknown parameters, to have an estimate of Σ, we exchange the unknown parameters by their conforming MLE's, given by

$$\hat{\Sigma} = \begin{bmatrix} \hat{V}_{\theta\theta} & \hat{V}_{\theta\lambda} \\ \hat{V}_{\lambda\theta} & \hat{V}_{\lambda\lambda} \end{bmatrix}^{-1}. \tag{23}$$

Using (15), approximately $100(1-\alpha)\%$ confidence intervals for θ and λ can be determined respectively, as

$$\hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\theta\theta}}\ , \qquad \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\lambda\lambda}}\ ,$$

Here, $Z_{\frac{\alpha}{2}}$ symbolizes the upper $\left(\frac{\alpha}{2}\right)^{th}$ percentile of the standard normal distribution.

VIII. ORDER STATISTICS

Let $Z_1, Z_2, \ldots, Z_k$ are independently identically distributed (i.i.d) ordered random variable selected form from VFWEx having parameters (θ, λ) in such a manner that $Z_{(1:k)} \le \ldots \le Z_{(k:k)}$. So, the density function of $Z(i:k)$, i=1, 2, 3, . . . , k is

$$g_{i:k}(z) = \frac{1}{Beta(i, n-i+1)} g(z,\Theta)\left[G(z,\Theta)\right]^{i-1}\left[1-G(z,\Theta)\right]^{k-i}. \tag{24}$$

Where $\Theta = (\theta, \lambda)$. And, the joint density of $((z_{i:k}),(z_{k:k}))$ is

$$g_{i:j:k}(z_i, z_j) = \psi\left[G(z_i)\right]^{i-1}\left[G(z_j)-G(z_i)\right]^{j-i-1}\left[1-G(z_j)\right]^{k-j} g(z_i)g(z_j).$$

Where

$$\psi = \frac{k!}{(i-1)!(j-i-1)!(k-j)!}.$$

The expressions, for the k^{th} order statistics as $Z_{(k)} = \max(Z_1, Z_2, \ldots, Z_k)$, median order statistics as Z_{m+1}, if $k = 2m+1$, and the 1^{st} order statistics as $Z_{(1)} = \min(Z_1, Z_2, \ldots, Z_k)$, are derived.

A. *Distribution of Maximum, Median and Minimum Order Statistics*

Let $Z_1,\ Z_2,\ ...\ ,\ Z_k$ are sampled randomly from VFWEx distribution with CDF provided in (10). Then, the density functions of the maximum, median and minimum order statistics are derived in (25)-(27), respectively. The density function for the k^{th} order statistic is derived as

$$g_{k:k}(z) = k\,g(z)\left[G(z)\right]^{k-1},$$

$$g_{k:k}(z) = k\left(\theta + \frac{\lambda}{z^{\lambda+1}}\right) e^{\left(\theta z - \frac{1}{z^{\lambda}}\right)} e^{-e^{\left(\theta z - \frac{1}{z^{\lambda}}\right)}} \left\{1 - e^{-e^{\left(\theta z - \frac{1}{z^{\lambda}}\right)}}\right\}^{k-1}. \tag{25}$$

The density function for the median order statistic is derived as

$$g_{m+1:k}(\tilde{z}) = \frac{(2m+1)!}{m!m!} g(\tilde{z})\left\{G(\tilde{z})\right\}^m \left\{1 - G(\tilde{z})\right\}^m,$$

$$g_{m+1:k}(\tilde{z}) = \frac{(2m+1)!}{m!m!}\left(\theta + \frac{\lambda}{z^{\lambda+1}}\right) e^{\left(\theta z - \frac{1}{z^{\lambda}}\right)} e^{-e^{\left(\theta z - \frac{1}{z^{\lambda}}\right)}} \left\{1 - e^{-e^{\left(\theta z - \frac{1}{z^{\lambda}}\right)}}\right\}^m \left\{e^{-e^{\left(\theta z - \frac{1}{z^{\lambda}}\right)}}\right\}^m. \tag{26}$$

The density function for the 1^{th} order statistic is derived as

$$g_{1:k}(z) = k\,g(z)\left[1 - G(z)\right]^{k-1}.$$

$$g_{1:k}(z) = k\left(\theta + \frac{\lambda}{z^{\lambda+1}}\right) e^{\left(\theta z - \frac{1}{z^{\lambda}}\right)} \left\{e^{-e^{\left(\theta z - \frac{1}{z^{\lambda}}\right)}}\right\}^k. \tag{27}$$

B. *The Joint Density Function of i^{th} and j^{th} Order Statistics*

The joint density function of the i^{th} and j^{th} order statistics observed from VFWEx distribution can be derived as

$$g_{i:j:k}(z_i, z_j) = \psi\left[G(z_i)\right]^{i-1}\left[G(z_j) - G(z_i)\right]^{j-i-1}\left[1 - G(z_j)\right]^{k-j} g(z_i)\,g(z_j).$$

Where,

$$\psi = \frac{k!}{(i-1)!(j-i-1)!(k-j)!}.$$

For special case: Let $i = 1$ and $j = k$, one may have the joint density of minimum and maximum order statistics as

$$g_{1:k:k}(z_1, z_k) = k(k-1)\left[e^{-e^{\left(\theta z_1 - \frac{1}{z_1^\lambda}\right)}} - e^{-e^{\left(\theta z_k - \frac{1}{z_k^\lambda}\right)}}\right]^{k-2}\left(\theta + \frac{\lambda}{z_1^{\lambda+1}}\right)e^{\left(\theta z_1 - \frac{1}{z_1^\lambda}\right)}\, e^{-e^{\left(\theta z_1 - \frac{1}{z_1^\lambda}\right)}}$$

$$\times\left(\theta + \frac{\lambda}{z_k^{\lambda+1}}\right)e^{\left(\theta z_k - \frac{1}{z_k^\lambda}\right)}\, e^{-e^{\left(\theta z_k - \frac{1}{z_k^\lambda}\right)}}.$$

IX. APPLICATION

To illustrate the VFWEx distribution, the analysis of a well-known and most frequently used real data set is presented. The goodness of fit result of the proposed model is compared with other modified of forms of Weibull model using Hannan-Quinn information criterion (HQIC), (AIC) [5] as well as Bayesian information criterion (BIC) [18] and log-likelihood. The goodness of fit result is compared with that of twelve other lifetime models such as: Weibull, Flexible Weibull Extension (FWEx), Exponentiated Weibull (EW), Exponentiated Flexible Weibull Extension (EFWEx), Exponential Flexible Weibull Extension (EFWEx), Transmuted Weibull (TW), Kumaraswamy Weibull (Ku-W), Modified Weibull (MW), New Modified Weibull (NMW), Additive Weibull (ADW), Beta Modified Weibull (BMW) and Kumaraswamy Flexible Weibull Extension (Ku-FWEx) distributions. The data represent the lifetimes of fifty devices obtained from Arset [1], and possess a bathtub shaped failure rate property. Numerous researcher have considered this data set, including Mudholkar and Srivastava [15], Sarhan and Zaindin [17], Silva et al. [19], Lai et al. [12] and Almalki and Yuan [6]. The data set is listed in table 1, table 2 provides the summary of the data listed in table 1. Finally, table 3 provides the Log-likelihood HQIC, AIC and BIC values of fitted models corresponding to Aarst data.

Table 1: Life time of 50 devices.

0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 1, 1, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86

Table 2: Summary of the Arset data.

Min	1st Quartile	Median	Mean	3rd Quartile	Max
0.10	9.50	47.00	44.61	80.50	86.00

Table 3: Goodness of fit result of the proposed and other competing models corresponding to Arset data

Dist.	Max. Likelihood Estimates	Log-likelihood	HQIC	AIC	BIC
VFWEx	$\hat{\theta}$**=0.0176,** $\hat{\lambda}$**=0.134**	**-91.66**	**188.81**	**187.33**	**191.20**
FWEx	$\hat{\alpha}$ = 0.0122, $\hat{\beta}$ = 0.7002	-250.81	507.12	505.62	509.44
W	$\hat{\alpha}$ =44.913, $\hat{\beta}$ = 0.949	-241.00	487.09	486.00	489.82
EW	$\hat{\alpha}$ =91.023, $\hat{\beta}$ =4.69, $\hat{\sigma}$ = 0.164	-235.92	479.80	477.85	483.58
EFWEx	$\hat{\alpha}$ =0.0147, $\hat{\beta}$ =0.0147, $\hat{\theta}$ = 4.22	-226.98	463.67	459.97	465.71
EFWEx	$\hat{\alpha}$ =0.015, $\hat{\beta}$ =0.381, $\hat{\lambda}$ = 0.076	-224.83	465.66	455.66	461.40
TW	$\hat{\alpha}$ =0.83, $\hat{\beta}$ =0.05, $\hat{\lambda}$ =-0.29	-243.56	403.19	493.13	498.93
Ku-W	$\hat{\alpha}$ =0.92, $\hat{\beta}$ =0.008, $\hat{a}$=0.85, $\hat{b}$ =3.00	-243.69	491.98	495.38	503.11
MW	$\hat{\beta}$ =0.062, $\hat{\gamma}$ =0.356, $\hat{\lambda}$ =0.023	-227.16	465.38	460.38	466.0
NMW	$\hat{\alpha}$ =0.071, $\hat{\beta}$ =0.070, $\hat{\gamma}$ =0.016, $\hat{\theta}$=0.59, $\hat{\lambda}$ =0.197	-212.90	435.05	435.85	445.4
ADW	$\hat{\alpha}$ =0.075, $\hat{\beta}$ =0.086, $\hat{\gamma}$ =0.477, $\hat{\theta}$=4.214,	-221.51	459.89	451.09	458.7
BMW	$\hat{\alpha}$ =0.013, $\hat{\beta}$ =0.082, $\hat{\gamma}$ =4.224	-220.80	459.97	451.67	461.2
Ku-FWEx	$\hat{\alpha}$ =0.0130, $\hat{\beta}$ =0.1270, $\hat{a}$=4.648, $\hat{b}$ =1.324	-226.67	465.94	461.34	468.99

X. CONCLUSION

A new lifetime distribution, by considering a linear scheme of the two logarithms of cumulative hazard functions, has been introduced and its statistical properties are studied. The proposed distribution is much flexible, so that it is able to model with increasing, unimodal model, modified unimodal or more prominently, bathtub shaped. It showed that the new flexible Weibull extended distribution fits certain data set better than Weibull distribution and its other well-known existing modifications. Statistical properties along with the estimation using maximum likelihood method are discussed. Reducing the number of parameters by fixing one parameter constant, still the proposed model provides better fit than other existing models.

Future framework includes MCMC approach with censored data, regression problems with covariates as well as parameter reduction are under study.

ACKNOWLEDGMENT

The authors are grateful to editor and anonymous referees for a careful checking of the details and for helpful comments that improved this article.

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R CODES FOR THE MODEL FITTING

Here, **a** is used for α , *l* is used for λ , and **pm** is used for proposed model

```
data=c(0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 1, 1, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67,
67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86)

# Proposed Model (pm) - Probability density function

pdf_pm <- function(par,x)

{
```

```
a= par[1]

l= par[2]

(a*x+(( l)/x^( l +1)))*(exp((a*x)-(1/x^l)))*exp(-((exp((a*x)-(1/x^l)))))

}

# Proposed Model (pm)   - Cumulative distribution function.

cdf_pm <- function(par,x)

{

a= par[1]

l= par[2]

1- exp(-((exp((a*x)-(1/x^l)))))

}

set.seed(0)

goodness.fit(pdf=pdf_pm,

cdf=cdf_pm,

starts =c(1,1), data = data,

method="SANN", domain=c(0,Inf),mle=NULL)
```

Biography

Zubair Ahmad S/O Wali Muhammad, research scholar at Quaid-i-Azam University, obtained his Master degree in Statistics in 2014 at University of Malakand and received his M. Phil. Degree in Statistics in 2017 at Quaid-i-Azam University, Islamabad, Pakistan. His research topic in M. Phil. was **"On Different Modifications of Weibull Distribution"** under the guidance & supervision of Dr. Zawar Hussain, Quaid-i-Azam University, Islamabad, Pakistan.