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Unit Generalized Quaternions in Spatial Kinematics

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Abstract- **After a review of some fundamental properties of the generalized quaternions, we apply a unit generalized quaternion to rotation in the 3-dimensional space $E^3_{\alpha\beta}$. Also, the angular velocity of this motion is obtained.**

***Keywords*: Generalized quaternion, quasi-orthogonal matrix, rotation.**

I. INTRODUCTION

The number system of quaternions is an extended one of the complex numbers. They were first described by W.R. Hamilton in 1843 and applied to mechanics in three-dimensional space. The algebra of quaternions is often denoted by H. It can also be given by the Clifford algebra classifications $Cl_{0,2}(R) \cong Cl^0_{3,0}(R)$. The algebra H holds a special place in analysis since, according to the Frobenius theorem, it is one of the only two finite-dimensional division rings containing the real numbers as a proper subring; the other being the complex numbers. Unit quaternions provide a convenient mathematical notation for representing orientations and rotations of objects in the three dimensional space. The unit quaternions can therefore be thought of as a choice of a group structure on the 3-sphere that gives the group Spin (3), which is isomorphic to SU(2) and also to the universal cover of SO(3). Kula and Yayli [1] showed that a unit split quaternion determines a rotation in semi-Euclidean space E^4_2. In [2], it is demonstrated how unit timelike split quaternions are used to perform rotations in the Minkowski 3-space E^3_1. Rotations in a complex 3-dimensional space are considered in [9] and applied to the treatment of the Lorentz transformation in special relativity.

A brief introduction of the generalized quaternions is provided in [4]. Recently, the work was done on the generalized quaternions by Jafari [5] and there is given some of their algebraic properties. In this paper, we briefly recall some fundamental properties of the generalized quaternions, and show that the set of all unit generalized quaternions with the group operation of quaternion multiplication is a Lie group of 3-dimension. A similar relation to the relationship between quaternions and rotations in the Euclidean space exists between generalized quaternions and rotations in the 3-space $E^3_{\alpha\beta}$. We demonstrate how a unit generalized quaternion is used to represent a rotation of a vector in 3-dimensional space $E^3_{\alpha\beta}$. The angular velocity of this motion is obtained. Finally, we give some examples for more clarification.

A. *Reel and split quaternions*

A real quaternion q is defined as

$$q = a_\circ + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

where $a_\circ, a_1, a_2$ and a_3 are real numbers and $1, \vec{i}, \vec{j}, \vec{k}$ of q may be interpreted as the four basic vectors of Cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = \vec{i}\,\vec{j}\,\vec{k} = -1,$$

$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = \vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}.$$

A quaternion q may be defined as a pair $(S_q, \vec{V}_q)$, where $S_q = a_\circ \in \mathrm{R}$ is scalar part and

$$\vec{V}_q = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \in \mathrm{R}^3$$

is the vector part of q. The quaternion product of two quaternions p and q is defined as

$$qp = S_q S_p - \langle \vec{V}_q, \vec{V}_p \rangle + S_q\vec{V}_p + S_p\vec{V}_q + \vec{V}_p \wedge \vec{V}_q$$

where "$\langle\, , \rangle$" and "$\wedge$" are the inner and vector products in R^3, respectively. The norm of a quaternion is given by the sum of the squares of its components:

$$N_q = a_\circ^2 + a_1^2 + a_2^2 + a_3^2\,, N_q \in \mathrm{R}.$$

It can also be obtained by multiplying the quaternion by its conjugate ($\bar{q} = S_q - \vec{V}_q$), in either order since a quaternion and of conjugated commute: $N_q = \bar{q}\,q = q\,\bar{q}$.
Every non-zero quaternion has a multiplicative inverse given by its conjugate divided by its norm: $q^{-1} = \bar{q} / N_q$. The quaternion algebra H is a normed division algebra, meaning that for any two quaternions p and q, $N_{pq} = N_p N_q$, and the norm of every non-zero quaternion is non-zero (and positive) and therefore the multiplicative inverse exists for any non-zero quaternion. Of course, as is well known, multiplication of quaternions is not commutative, so that in general for any two quaternions p and q, $pq \neq qp$. This can have subtle ramifications, for example: $(pq)^2 = pq\,pq \neq p^2q^2$. For detailed information about real quaternions, we refer the reader to [5].
The algebra H' of split quaternions is defined as the four-dimensional vector space over R having a basis $\{1, \vec{i}, \vec{j}, \vec{k}\}$ with the following properties;

$$\vec{i}^2 = -1, \ \vec{j}^2 = \vec{k}^2 = 1,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = -\vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}.$$

The quaternion product of two split quaternions p and q is defined as

$$qp = S_q S_p + \langle \vec{V}_q, \vec{V}_p \rangle_l + S_q\vec{V}_p + S_p\vec{V}_q + \vec{V}_p \wedge_l \vec{V}_q$$

where "$\langle\, , \rangle_l$" and "$\wedge_l$" are Lorentzian inner and vector products, respectively. It is clear that H and H' are associative and non-commutative algebras [2].

Definition 1. Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathrm{R}^3$. If $\alpha, \beta \in \mathrm{R}^+$, the generalized inner product of $\vec{u}$ and $\vec{v}$ is defined by

$$g(\vec{u}, \vec{v}) = \alpha\, u_1\, v_1 + \beta\, u_2\, v_2 + \alpha\beta\, u_3\, v_3,$$

It could be written

$$g(\vec{u}, \vec{v}) = u^T \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix} v = u^T G\, v.$$

Also, if $\alpha > 0, \beta < 0$, $g(\vec{u}, \vec{v})$ is called the generalized Lorentzian inner product. We put $\mathrm{E}^3_{\alpha\beta} = (\mathrm{R}^3, g(,))$, The vector product in $\mathrm{E}^3_{\alpha\beta}$ is defined by

$$\vec{u} \times \vec{v} = \begin{vmatrix} \beta\vec{i} & \alpha\vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$= \beta(u_2v_3 - u_3v_2)\vec{i} + \alpha(u_3v_1 - u_1v_3)\vec{j} + \alpha\beta(u_1v_2 - u_2v_1)\vec{k},$$

where $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \beta\vec{i}$, $\vec{k} \times \vec{i} = \alpha\vec{j}$.

Special cases:

1. If $\alpha = \beta = 1$, then $E^3_{\alpha\beta}$ is an Euclidean 3-space E^3.
2. If $\alpha = 1$, $\beta = -1$, then $E^3_{\alpha\beta}$ is a Minkowski 3-space E^3_1 [2].

Proposition 1. For $\alpha, \beta \in R^+$, the inner and vector products satisfy the following properties;

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.
2. $g(\vec{u} \times \vec{v}, \vec{w}) = g(\vec{v} \times \vec{w}, \vec{u}) = g(\vec{u} \times \vec{w}, \vec{v})$.
3. $\vec{u} \times (\vec{v} \times \vec{w}) = g(\vec{u} \times \vec{w})\vec{v} - g(\vec{u}, \vec{v})\vec{w}$.

Definition 2. A matrix $A_{3\times 3}$ is called a quasi-orthogonal matrix if $A^T \varepsilon A = \varepsilon$ and $\det A = 1$ where

$$\varepsilon = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}$$

and $\alpha, \beta \in R - \{0\}$. The set Q(3) of all quasi-orthogonal matrices with the operation of matrix multiplication constitutes a group, called rotation group, in the 3-space $E^3_{\alpha\beta}$.

Definition 3. Consider rotational motion of a rigid body, *i.e.*, a rigid body motion leaving one point fixed. Let $r(0)$ be the position of a point in the body at $t = 0$, $r(t)$ its position at time t, and $R(t)$ the rotation operator which carries $r(0)$ to $r(t)$

$$r(t) = R(t)\, r(0).$$

$R(t)$ thus traces the orientation of the rigid body in configuration space SO(3). The velocity of the point is $\dot{r} = \dot{R}\, r(0) = \dot{R} R^{-1} r$ or $\dot{r} = \Omega r$, $\Omega = \dot{R} R^{-1}$. The operator Ω is the angular velocity operator [5].

B. Generalized quaternions algebra

A generalized quaternion q is an expression of form

$$q = a_0 + a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units which satisfy the equalities

$$\vec{i}^2 = -\alpha, \quad \vec{j}^2 = -\beta, \quad \vec{k}^2 = -\alpha\beta,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = \beta\vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \alpha\vec{j} = -\vec{i}\vec{k}, \quad \alpha, \beta \in R.$$

The set of all generalized quaternions are denoted by $H_{\alpha\beta}$. A generalized quaternion q is a sum of a scalar and a vector, called scalar part, $S_q = a_0$, and vector part $\vec{V}_q = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \in E^3_{\alpha\beta}$. Therefore, $H_{\alpha\beta}$ is form a 4-dimensional real space which contains the real axis R and a 3-dimensional real linear space $E^3_{\alpha\beta}$, so that, $H_{\alpha\beta} = R \oplus E^3_{\alpha\beta}$.

If $q = (a_0, \vec{V}_q)$ and $p = (b_0, \vec{V}_p)$ are two quaternions, their sum is defined as

$$q + p = (a_0 + b_0,\ \vec{V}_q + \vec{V}_p)$$

and their product (non-commutative) as

$$qp = (a_0 b_0 - g(\vec{V}_q, \vec{V}_p),\ a_0\vec{V}_p + b_0\vec{V}_q + \vec{V}_p \times \vec{V}_q),$$

here "g(,)" and "×" are the inner and vector products in $E^3_{\alpha\beta}$, respectively. The conjugate quaternion of q is defined as $\bar{q} = (a_0, -\vec{V}_q)$ and the length or norm as

$$N_q = q\bar{q} = \bar{q}q = a_\circ^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 \in R.$$

Note that $N_{qp} = N_q N_p$. Every non-zero quaternion has a multiplicative inverse given by its conjugate divided by its norm: $q^{-1} = \overline{q} / N_q$. The generalized quaternion with a norm of one, $N_q = 1$, is a unit generalized quaternion.
If a generalized quaternion is looked at as a four-dimensional vector, the generalized quaternion product can be described by a matrix-vector product as

$$qp = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

II. ROTATION OPERATORS IN $\mathrm{H}_{\alpha\beta}$

In Euclidean 3-space, there are a lot of methods used to represent rotations like orthogonal matrices, Euler angles and quaternions. Quaternionic representation is the most useful one for the purpose. If we compare to orthogonal matrices, there are some constraints as each column of an orthogonal matrix must be unit vector and they must be perpendicular to each other. These constraints make it difficult to construct an orthogonal matrix using nine numbers. But we can construct easily a rotation orthogonal matrix using a unit quaternion. That is, only four numbers are enough to represent a rotation such that there is only one constraint which is that the norm of the quaternion must be equal to one. Here, we show that every unit generalized quaternion represents a rotation in the 3-space.
Let a Lie group G be given, to its every element q the mapping $C_q : G \to G$, $x \to qxq^{-1}$ for all $x \in G$ is assigned. The mapping C_q is a differentiable isomorphism (inner automorphism) of the group G, and the differential of C_q at the point e, *i.e.* $(dC_q)_e = adq$ maps $T_G(e)$ into $T_G(e)$. Then, the mapping adq is called the adjoint representation of the group G [8].

If we associate the generalized quaternion $x = (0, \vec{x})$ with the three-dimensional vector $\vec{x}$ and define the operation, with the unit generalized quaternion q, as

$$adq(x) = x' = qxq^{-1} = qx\overline{q}, \tag{1}$$

then this transformation, from x to x', represents a rotation in 3-space $\mathrm{E}^3_{\alpha\beta}$. The resulting quaternion x' is a vectorial quaternion with the same length as x, since

$$N_{x'} = N_q N_{xq^{-1}} = N_q N_x N_{q^{-1}} = N_x.$$

Theorem 3. For any unit generalized quaternion *q*, the matrix corresponding to map adq is

$$adq = \begin{bmatrix} a_0^2 + \alpha a_1^2 - \beta a_2^2 - \alpha\beta a_3^2 & 2\beta(a_1 a_2 - a_0 a_3) & 2\beta(\alpha a_1 a_3 + a_0 a_2) \\ 2\alpha(a_1 a_2 + a_0 a_3) & a_0^2 - \alpha a_1^2 + \beta a_2^2 - \alpha\beta a_3^2 & 2\alpha(\beta a_2 a_3 - a_0 a_1) \\ 2(\alpha a_1 a_3 - a_0 a_2) & 2(a_0 a_1 + \beta a_2 a_3) & a_0^2 - \alpha a_1^2 - \beta a_2^2 + \alpha\beta a_3^2 \end{bmatrix}.$$

Special cases:

1) For the case $\alpha = \beta = 1$, we get adq for real quaternion H, denote it by *M*. *M* is a orthogonal matrix, then the map adq corresponds to a rotation in E^3. If we take the rotation axis to be $S = (s_1, s_2, s_3)$, then we have

$$M = I_3 + S\sin\theta + (1-\cos\theta)S^2,$$

where S is a skew-symmetric matrix,

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}.$$

2) For the case $\alpha=1,\ \beta=-1,$ we get adq for split quaternion H', denote it by N. N is a semi-orthogonal matrix, then the map adq corresponds to a rotation in 3-Minkowski space E_1^3. If we take the rotation axis to be $C=(c_1,c_2,c_3),$ then we have

$$N=I_3+\sinh\gamma\, C+(-1+\cosh\gamma)C^2,$$

where C is a skew-symmetric matrix, *i.e.* $C^T=-\kappa C\kappa$ and

$$C=\begin{bmatrix}0 & c_3 & -c_2\\ c_3 & 0 & -c_1\\ -c_2 & c_1 & 0\end{bmatrix},\quad \kappa=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\ [1].$$

Theorem 4. For every $q\in G$ and $\alpha,\beta\in\mathrm{R}$, adq is a quasi-orthogonal matrix.

Theorem 5. Let q be a unit generalized quaternion and $\alpha,\beta\in\mathrm{R}^+$, then the matrix adq can be written as

$$adq=I_3+\sin\varphi\,\mathrm{S}+(1-\cos\varphi)\mathrm{S}^2,$$

where $\cos\frac{\varphi}{2}=a_0,\ s_i\sin\frac{\varphi}{2}=a_i,\ i=1,2,3.$

$$S=\begin{bmatrix}0 & -\beta s_3 & \beta s_2\\ \alpha s_3 & 0 & -\alpha s_1\\ -s_2 & s_1 & 0\end{bmatrix}$$

is a skew-symmetric matrix, *i.e.*, $S^T\varepsilon=-\varepsilon S.$

Proof: Every unit generalized quaternion $q=a_0+a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ can be written in polar form

$$q=\cos\frac{\phi}{2}+\vec{S}\sin\frac{\phi}{2},$$

then the matrix adq can be written as

$$adq=\begin{bmatrix}\cos^2\frac{\phi}{2}+(\alpha s_1^2-\beta s_2^2-\alpha\beta s_3^2)\sin^2\frac{\phi}{2} & 2\beta(s_1s_2\sin^2\frac{\phi}{2}-s_3\cos\frac{\phi}{2}\sin\frac{\phi}{2}) & 2\beta(\alpha s_1s_3\sin^2\frac{\phi}{2}+s_2\cos\frac{\phi}{2}\sin\frac{\phi}{2})\\ 2\alpha(s_1s_2\sin^2\frac{\phi}{2}+s_3\cos\frac{\phi}{2}\sin\frac{\phi}{2}) & \cos^2\frac{\phi}{2}+(-\alpha s_1^2+\beta s_2^2-\alpha\beta s_3^2)\sin^2\frac{\phi}{2} & 2\alpha(\beta s_2s_3\sin^2\frac{\phi}{2}+s_1\cos\frac{\phi}{2}\sin\frac{\phi}{2})\\ 2(\alpha s_1s_3\sin^2\frac{\phi}{2}-s_2\cos\frac{\phi}{2}\sin\frac{\phi}{2}) & 2(\beta s_2s_3\sin^2\frac{\phi}{2}+s_1\cos\frac{\phi}{2}\sin\frac{\phi}{2}) & \cos^2\frac{\phi}{2}+(-\alpha s_1^2-\beta s_2^2+\alpha\beta s_3^2)\sin^2\frac{\phi}{2}\end{bmatrix}$$

$$=I_4+\begin{bmatrix}(\alpha s_1^2-\beta s_2^2-\alpha\beta s_3^2-1)\sin^2\frac{\phi}{2} & 2\beta s_1s_2\sin^2\frac{\phi}{2}-\beta s_3\sin\phi & 2\alpha\beta s_1s_3\sin^2\frac{\phi}{2}+\beta s_2\sin\phi\\ 2\alpha s_1s_2\sin^2\frac{\phi}{2}+\alpha s_3\sin\phi & (-\alpha s_1^2+\beta s_2^2-\alpha\beta s_3^2-1)\sin^2\frac{\phi}{2} & 2\alpha\beta s_2s_3\sin^2\frac{\phi}{2}+\alpha s_1\sin\phi\\ 2\alpha s_1s_3\sin^2\frac{\phi}{2}-s_2\sin\phi & 2\beta s_2s_3\sin^2\frac{\phi}{2}+s_1\sin\phi & (-\alpha s_1^2-\beta s_2^2+\alpha\beta s_3^2-1)\sin^2\frac{\phi}{2}\end{bmatrix}$$

with used of $2\sin^2\frac{\phi}{2}=1-\cos\phi$ and $\alpha s_1^2+\beta s_2^2+\alpha\beta s_3^2=1$, we have

$$\mathrm{M}=I_3+\sin\phi\begin{bmatrix}0 & -\beta s_3 & \beta s_2\\ \alpha s_3 & 0 & -\alpha s_1\\ -s_2 & s_1 & 0\end{bmatrix}+(1-\cos\phi)\begin{bmatrix}-\alpha\beta s_3^2-\beta s_2^2 & \beta s_1s_2 & \alpha\beta s_1s_3\\ \alpha s_1s_2 & -\alpha s_1^2-\alpha\beta s_3^2 & \alpha\beta s_2s_3\\ \alpha s_1s_3 & \beta s_2s_3 & -\alpha s_1^2-\beta s_2^2\end{bmatrix},$$

so, proof is complete.

Example. Let $\vec{x}\in\mathrm{E}_{\alpha\beta}^3$ such that $\vec{x}=2\vec{i}+0\vec{j}+\vec{k}$. We want to rotate point $\vec{x}$ about the unit vector $\vec{s}=\frac{1}{\sqrt{2\alpha}}\vec{i}+\frac{1}{\sqrt{2\beta}}\vec{j}$ at an angle of $\varphi=\frac{\pi}{3}$. Using theorem 3, we find a unit generalized quaternion q that will accomplish such a rotation.

$$\begin{aligned} q &= \cos\frac{\phi}{2} + \vec{S}\sin\frac{\phi}{2} \\ &= \cos\frac{\pi/3}{2} + \left(\frac{1}{\sqrt{2\alpha}}, \frac{1}{\sqrt{2\beta}}, 0\right)\sin\frac{\pi/3}{2} \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{2\alpha}}\vec{i} + \frac{1}{2\sqrt{2\beta}}\vec{j}. \end{aligned}$$

Now we have defined a unit generalized quaternion q such that $q\,\vec{x}\,q^{-1} = \vec{x}\,'$ where $\vec{x}\,'$ is the point $\vec{x}$ rotated about vector $\vec{s}$ at an angle of $\frac{\pi}{3}$. Now that q is known, the operator adq is applied.

$$\begin{aligned} \vec{x}\,' &= q\,\vec{x}\,\overline{q} \\ &= (\frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{2\alpha}}\vec{i} + \frac{1}{2\sqrt{2\beta}}\vec{j})(2\vec{i} + 0\vec{j} + \vec{k}) \\ &\quad (\frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{2\alpha}}\vec{i} - \frac{1}{2\sqrt{2\beta}}\vec{j}) \\ &= (\frac{3}{2} - \frac{\sqrt{3\beta}}{2\sqrt{2}}, \frac{\sqrt{\alpha}}{2\sqrt{\beta}} - \frac{\sqrt{3\alpha}}{2\sqrt{2}}, \frac{1}{2} - \frac{\sqrt{3}}{\sqrt{2\beta}}). \end{aligned}$$

Corollary: Let $q = \cos\frac{\phi}{2} + \vec{S}\sin\frac{\phi}{2}$ be a unit generalized quaternion. If $\alpha, \beta \in \mathrm{R}^+$, the linear map $\varphi(\omega) = q\omega q^{-1}$ represents a rotation of the original vector ω by an angle ϕ around the axis $\vec{S}$ in 3-space $\mathrm{E}^3_{\alpha\beta}$. Also, if $\alpha > 0, \beta < 0$, the above map represents a rotation in Lorentzian manner (boost).

III. ANGULAR VELOCITY OF THE MOTION

Suppose that the orientation of a body is represented as a quaternion rotation so that $x' = qxq^{-1}$ where x is a vector in fixed space coordinates and x' is a vector in body coordinates. By differentiational of the rotational transformation (1) as in

$$\dot{x}' = \dot{q}\,x\,\overline{q} + q\,x\,\dot{\overline{q}},$$

expressing it in terms of fixed coordinates gives

$$\begin{aligned} \dot{x}' &= \dot{q}(\overline{q}q)\,x\,(\overline{q}q)\,\overline{q} + q\,(\overline{q}q)x\,(\overline{q}q)\dot{\overline{q}}, \\ &= \dot{q}\,\overline{q}\,x' + x'\,q\,\dot{\overline{q}}. \end{aligned}$$

The scalar part of $\dot{q}\,\overline{q}$ happens to be zero because q is a unit generalized quaternion, so

$$\dot{q}\,\overline{q} = -\overline{(\dot{q}\,\overline{q})} = q\,\dot{\overline{q}},$$

applying this into the above gives

$$\dot{x}' = \dot{q}\,\overline{q}\,x' - x'\,\dot{q}\,\overline{q}.$$

which is just the antisymmetric product of $\dot{q}\,\overline{q}$ and x'. So the velocity $\dot{x}'$ has a zero scalar part and a vectorial part, $2\,\vec{\omega} \times x'$ where $\dot{q}\,\overline{q} = (0, \vec{\omega})$. We conclude that the angular velocity $w = \vec{\omega}$ expressed in the space fixed reference in terms of the Euler parameters q and time derivatives as follows

$$w = 2\dot{q}\,\overline{q}$$

or

$$\begin{pmatrix} 0 \\ w \end{pmatrix} = 2\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ -\alpha a_1 & a_0 & -\beta a_3 & \beta a_2 \\ -\beta a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ -\alpha\beta a_3 & -a_2 & a_1 & a_0 \end{bmatrix}\begin{bmatrix} \dot{a}_0 \\ \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \end{bmatrix}.$$

Special Case:

If $\alpha = \beta = 1$, then w is the angular velocity for real quaternion rotation [7].

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