MAYFEB Journal of Mathematics
Vol 4 (2016) - Pages 1-6

Identities Involving the Product of k-Fibonacci Numbers and k-Lucas Numbers

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***Abstract*-This paper represents the investigation of products of k-Fibonacci and k-Lucas numbers. It also reveals some generalized identities on the product of k-Fibonacci and k-Lucas numbers.The purpose is to establish connection formulas between them with the help of Binet`s formula.**

***Keywords*: k-fibonacci numbers, k-lucas numbers, product of k-fibonacci and k-lucas numbers, binet's formula.**

I. INTRODUCTION

The alarmingly vast world integer sequence seems to feel pride in having two great stars the Fibonacci and the Lucas sequence. Their magnetic effect has drawn both amateurs and professional mathematicians for the research work for the so far hidden miracles in the sphere of mathematics. Their surprisingly multidimensional applications keep on charming the scholars' interest in search for some more applications even in interested places. Comprehensive works of eminent mathematicians like Koshy[11] and Vajda[10] reflect that the Fibonacci and the Lucas numbers have engripped the interest of pretty good number of scholars inclined towards the miracles falling in the circumference of mathematics. In Fibonacci sequence each term is the sum of the two previous terms beginning with the initial values $F_0 = 0$ and $F_1 = 1$ and the ratio of two consecutive Fibonacci numbers converges to the Golden mean $\emptyset = \frac{1+\sqrt{5}}{2}$.

The Fibonacci sequence [10] is defined as,

$$F_0 = 0,\ F_1 = 1,$$

$$F_n = F_{n-1} + F_{n-2},\ \ n\geq 2. \tag{1}$$

The Lucas sequence [10] is defined as,

$$L_0=2\ ,\ L_1 =1,$$

$$L_n=L_{n-1} + L_{n-2},\ \ n\geq 2 \tag{2}$$

The k-Fibonacci number defined by Falcon and Plaza [8] for any real k as follows:

$$F_{k,0} = 0,\ F_{k,1} = 1,$$

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}\ . \tag{3}$$

The first few terms of this sequence is

$$\{0, 1, k, k^2+1, k^2+2,\dots\}.$$

If $k = 1$,

The classical Fibonacci is obtained:

$$F_0 = 0,\ F_1 = 1,$$

$$F_{n+1} = F_n + F_{n-1},\ n \geq 1 \tag{4}$$

$$\{F_n\}_{n\in N} = \{0, 1, 1, 2, 3, 5, 8\dots.\}.$$

If k =2, the Pell sequence [5] is obtained :

$P_0 = 0$, $P_1=1$,

$P_{n+1} = 2P_n + P_{n-1}$, $n\geq 1$ (5)

$\{P_n\}_{n\in N} = \{ 0, 1, 2, 5, 12, 29, 70,\ldots\}$.

In a similar way, k-Lucas sequence is defined as,

$L_{k,o}= 2$, $L_{k,1}= k$,

$L_{k,n+1} = kL_{k,n} + L_{k,n-1}$, $n\geq 1$, $k\geq 1$. (6)

The first few terms of this sequence are $\{2, k, k^2+2, k^3+3,\ldots\}$.

If k =1, the classical Lucas sequence is obtained: {2, 1, 3, 4, 7, 11, 18,…}.

If k =2, the Pell-Lucas sequence [6] is obtained: {2, 2, 6, 14, 34, 82,…}.

Binet`s formula [10] for k-Fibonacci number and k-Lucas number [2- 6] are given by,

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \quad (7)$$

$$L_{k,n} = r_1^n + r_2^n.$$

Where r_1 and r_2 are roots of characteristic equation $r^2- kr-1 = 0$, which are given by,

$$r_1 = \frac{k+\sqrt{k^2+4}}{2}, r_2 = \frac{k-\sqrt{k^2+4}}{2}$$

$$r_1 + r_2 = k,$$

$$r_1 . r_2 = -1,$$

$$r_1 - r_2 = \sqrt{k^2+4}.$$

Identities of these are given in various literatures in various forms. This paper presents connection formulas between k-Fibonacci and k- Lucas number through application of Binet`s formula and induction method.

II. THEOREMS RELATED TO PRODUCT OF K-FIBONACCI AND K-LUCAS NUMBERS

A. *Product of k-Fibonacci and k-Lucas Number*

Theorem 2.1. $F_{k,2n}. L_{k,2n-1} = F_{k,4n-1} - 1$,where $n \geq 1$.

Proof:

$$F_{k,2n}.L_{k,2n-1}= \left[\frac{r_1^{2n}-r_2^{2n}}{r_1-r_2}\right].[r_1^{2n-1}+r_2^{2n-1}].$$

$$= \frac{1}{r_1-r_2}[r_1^{4n-1}+r_1^{2n}.r_2^{2n-1}-r_1^{2n-1}r_2^{2n}-r_2^{4n-1}]$$

$$= \frac{r_1^{4n-1}-r_2^{4n-1}}{r_1-r_2} - \frac{(r_1r_2)^{2\,n}}{r_1-r_3}[r_2^{-1}-r_1^{-1}]$$

$$= F_{k,4n-1} + (-1)^{2n}\left[\frac{1}{r_1r_2}\right]$$

$$= F_{k,4n-1} - 1. \quad (8)$$

Theorem 2.2. $F_{k,2n-1}$ $L_{k,2n} = F_{k,4n-1} + 1$, where n ≥ 1.

Proof:

$$F_{k,2n-1}L_{k,2n} = \left[\frac{r_1^{2n-1} - r_2^{2n-1}}{r_1 - r_2}\right].[r_1^{2n} + r_2^{2n}]$$

$$= \frac{1}{r_1-r_2}[r_1^{4n-1} + r_1^{2n-1}.r_2^{2n} - r_1^{2n}.r_2^{2n-1} - r_2^{4n-1}]$$

$$= \frac{r_1^{4n-1}-r_2^{4n-1}}{r_1-r_2} + \frac{\{r_1.r_2\}^{2n}}{r_1-r_2}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$= F_{k,4n-1} + \frac{(-1)^{2n}}{r_1-r_2}\left[\frac{r_2-r_1}{r_1r_2}\right]$$

$$= F_{k,4n-1} + 1. \quad (9)$$

Theorem 2.3. $F_{k,2n-1}$ $L_{k,2n+2} = F_{k,4n+1} + (k^2 + 1)$, where n ≥ 1.

Proof:

$$F_{k,2n-1}\ L_{k,2n+2} = \left[\frac{r_1^{2n-1}-r_2^{2n-1}}{r_1-r_2}\right].[r_1^{2n+2} + r_2^{2n+2}].$$

$$= \frac{1}{r_1-r_2}[r_1^{4n+1} + r_1^{2n-1}.r_2^{2n+2} - r_1^{2n+2}.r_2^{2n-1} - r_2^{4n}]$$

$$= \frac{r_1^{4n+1}-r_2^{4n+1}}{r_1-r_2} + \frac{\{r_1.r_2\}^{2n}}{r_1-r_2}\left[\frac{r_2^2}{r_1} - \frac{r_1^2}{r_2}\right]$$

$$= F_{k,4n+1} + \frac{(-1)^{2n}}{r_1-r_2}\left[\frac{r_2^3-r_1^3}{r_1r_2}\right]$$

$$= F_{k,4n+1} + \left[\frac{r_1^3-r_2^3}{r_1-r_2}\right]$$

$$= F_{k,4n+1} + \left[\frac{(r_1-r_2)(r_1^2+r_2^2+r_1r_2)}{r_1-r_2}\right]$$

$$= F_{k,4n+1} + [r_1^2 + r_2^2 - 1]$$

$$= F_{k,4n+1} + [L_{k,2} - 1]$$

$$= F_{k,4n+1} + [k^2 + 2 - 1]$$

$$= F_{k,4n+1} + [k^2 + 1]. \quad (10)$$

Theorem 2.4. $F_{k,2n+1}$ $L_{k,2n-1} = F_{k,4n} - k$, where n ≥ 1.

Proof:

$$F_{k,2n+1}\ L_{k,2n-1} = \left[\frac{r_1^{2n+1}-r_2^{2n+1}}{r_1-r_2}\right].[r_1^{2n-1} + r_2^{2n-1}]$$

$$= \frac{1}{r_1-r_2}[r_1^{4n} + r_1^{2n+1}.r_2^{2n-1} - r_1^{2n+1}.r_2^{2n-1} - r_2^{4n}]$$

$$= \frac{r_1^{4n}-r_2^{4n}}{r_1-r_2} + \frac{\{r_1.r_2\}^{2n}}{r_1-r_2}\left[\frac{r_1}{r_2} - \frac{r_2}{r_1}\right]$$

$$= F_{k,4n} + \frac{(-1)^{2n}}{r_1-r_2}\left[\frac{r_1^2-r_2^2}{r_1r_2}\right]$$

$$= F_{k,4n} + \frac{(-1)^{2n}}{r_1-r_2}\left[\frac{(r_1-r_2)(r_1+r_2)}{r_1r_2}\right]$$

$$= F_{k,4n} - k. \quad (11)$$

Theorem 2.5. $F_{k,2n+1}$ $L_{k,2n+1} = F_{k,4n+2}$, where n ≥ 1.

Proof:

$$F_{k,2n+1} L_{k,2n+1} = \left[\frac{r_1^{2n+1} - r_2^{2n+1}}{r_1 - r_2}\right].[r_1^{2n+1} + r_2^{2n+1}]$$

$$= \frac{1}{r_1 - r_2}[r_1^{4n+2} + r_1^{2n+1}.r_2^{2n+1} - r_1^{2n+1}.r_2^{2n+1} - r_2^{4n+2}]$$

$$= \frac{r_1^{4n+2} - r_2^{4n+2}}{r_1 - r_2}$$

$$= F_{k,4n+2}. \qquad (12)$$

Theorem 2.6. $F_{k,2n+1}$ $L_{k,2n+2} = F_{k,4n+3} + 1$, where n ≥ 1.

Proof:

$$F_{k,2n+1} L_{k,2n+2} = \left[\frac{r_1^{2n+1} - r_2^{2n+1}}{r_1 - r_2}\right].[r_1^{2n+2} + r_2^{2n+2}].$$

$$= \frac{1}{r_1 - r_2}[r_1^{4n+3} + r_1^{2n+1}.r_2^{2n+2} - r_1^{2n+2}.r_2^{2n+1} - r_2^{4n+3}]$$

$$= \frac{r_1^{4n+3} - r_2^{4n+3}}{r_1 - r_2} + \frac{(r_1.r_2)^{2n}}{r_1 - r_2}[r_1 r_2^2 - r_1^2 r_2]$$

$$= F_{k,4n+3} + \frac{(-1)^{2n}}{r_1 - r_2}[r_1 - r_2]$$

$$= F_{k,4n+3} + 1 \qquad (13)$$

Theorem 2.7. $F_{k,2n+2}$ $L_{k,2n-1} = F_{k,4n+1}$- (k^2+1) , where n ≥ 1.

Proof:

$$F_{k,2n+2} L_{k,2n-1} = \left[\frac{r_1^{2n+2} - r_2^{2n+2}}{r_1 - r_2}\right].[r_1^{2n-1} + r_2^{2n-1}].$$

$$= \frac{1}{r_1 - r_2}[r_1^{4n+1} + r_1^{2n+2}.r_2^{2n-1} - r_1^{2n-1}.r_2^{2n+2} - r_2^{4n+1}]$$

$$= \frac{r_1^{4n+1} - r_2^{4n+1}}{r_1 - r_2} + \frac{(r_1 r_2)^{2n}}{r_1 - r_2}\left[\frac{r_1^2}{r_2} - \frac{r_2^2}{r_1}\right]$$

$$= F_{k,4n+1} + \frac{(-1)^{2n}}{r_1 - r_2}\left[\frac{r_2^3 - r_1^3}{r_1 r_2}\right]$$

$$= F_{k,4n+1} - \left[\frac{(r_1 - r_2)(r_1^2 + r_2^2 + r_1 r_2)}{r_1 - r_2}\right]$$

$$= F_{k,4n+1} - [L_{k,2} - 1]$$

$$= F_{k,4n+1} - [k^2 + 2 - 1]$$

$$= F_{k,4n+1} - [k^2 + 1]. \qquad (14)$$

Theorem 2.8. $(k^2+4)\ F_{k,2n+1}\ F_{k,2n+2} = L_{k,4n+1} + k$, where $n \geq 1$.

Proof:

$$F_{k,2n+1}\ F_{k,2n+2} = \left[\frac{r_1^{2n+1}-r_2^{2n+1}}{r_1-r_2}\right].\left[\frac{r_1^{2n+2}-r_2^{2n+2}}{r_1-r_2}\right].$$

$$= \frac{[r_1^{4n+3}-r_1^{2n+1}.r_2^{2n+2}-r_1^{2n+2}.r_2^{2n+1}+r_2^{4n+3}]}{(r_1-r_2)^2}$$

$$= \frac{L_{k,4n+3}-(r_1r_2)^{2n}[r_1r_2^n+r_1^2r_2]}{(r_1-r_2)^2}$$

$$= \frac{L_{k,4n+3}+[r_1+r_2]}{(r_1-r_2)^2}$$

$$= \frac{L_{k,4n+3}+k}{k^2+4}$$

$$= (k^2+4)\ F_{k,2n+1}\ F_{k,2n+2}$$

$$= L_{k,4n+1} + k. \qquad (15)$$

III. THEOREMS RELATED TO GENERALIZED IDENTITIES ON THE K-FIBONACCI AND K-LUCAS NUMBERS

A. Generalized Identities on the Products of k- Fibonacci and k-Lucas

Theorem 3.1 $F_{k,m}.L_{k,n+2m} = F_{k,n+3m} - (-1)^m F_{k,n+m}$, where $n \geq 1, m \geq 0$.

Proof:

$$F_{k,m}.L_{k,n+2m} = \left[\frac{r_1^m-r_2^m}{r_1-r_2}\right].[r_1^{n+2m}+r_2^{n+2m}]$$

$$= \frac{1}{r_1-r_2}[r_1^{n+3m}+r_1^m.r_2^{n+2m}-r_1^{n+2m}.r_2^m-r_2^{n+3m}]$$

$$= \frac{r_1^{n+3m}-r_2^{n+3m}}{r_1-r_2} + \frac{\{r_1.r_2\}^m}{r_1-r_2}[r_2^{n+m}-r_1^{n+m}]$$

$$= F_{k,n+3m} - (-1)^m F_{k,n+m} \qquad (16)$$

Theorem 3.2. $F_{k,2m}.L_{k,2n} = F_{k,2n+2m} + F_{k,2m-2n}$, where $n \geq 1, m \geq 0$.

Proof:

$$F_{k,2m}.L_{k,2n} = \left[\frac{r_1^m-r_2^m}{r_1-r_2}\right].[r_1^{2n}+r_2^{n+2n}]$$

$$= \frac{1}{r_1-r_2}[r_1^{2n+2m}+r_1^{2m}.r_2^{2n}-r_1^{2n}.r_2^{2m}-r_2^{2m+2n}]$$

$$= \frac{r_1^{2m+2n}-r_2^{2m+2n}}{r_1-r_2} + \frac{(r_1r_2)^{2n}}{r_1-r_2}[r_1^{2m-2n}-r_2^{2m-2n}]$$

$$= F_{k,2n+2m} + F_{k,2n-2m} \qquad (17)$$

Theorem 3.3. $F_{k,m}.L_{k,2n+2m} = F_{k,2n+3m} + (-1)^m F_{k,2n+m}$, where $n \geq 1, m \geq 0$.

Proof:

$$F_{k,m}.L_{k,2n+2m} = \left[\frac{r_1^m-r_2^m}{r_1-r_2}\right].[r_1^{2n+2m}+r_2^{2n+2m}]$$

$$= \frac{1}{r_1-r_2}[r_1^{2n+3m}+r_1^m.r_2^{2n+2m}-r_2^m.r_1^{2n+2m}-r_2^{2n+3m}]$$

$$= \frac{r_1^{2n+3m}-r_2^{2n+3m}}{r_1-r_2} + \frac{\{r_1.r_2\}^m}{r_1-r_2}[r_1^{2n+m}-r_2^{2n+m}]$$

$$= F_{k,2n+3m} + (-1)^m F_{k,2n+m} \qquad (18)$$

Theorem 3.4. $F_{k,n+2m}.L_{k,2n+m} = F_{k,n+3m} + (-1)^m F_{k,2n+m}$, where $n \geq 1, m \geq 0$.

Proof:

$$F_{k,n+2m}.L_{k,2n+m} = \left[\frac{r_1^{n+2m} - r_2^{n+2m}}{r_1 - r_2}\right].\left[r_1^{2n+m} + r_2^{2n+m}\right]$$

$$= \frac{1}{r_1 - r_2}\left[r_1^{3n+3m} + r_1^{n+2m}.r_2^{2n+m} - r_2^{n+2m}.r_1^{2n+m} - r_2^{3n+3m}\right]$$

$$= \frac{r_1^{3n+3m} - r_2^{3n+3m}}{r_1 - r_2} - \frac{\{r_1.r_2\}^m (r_1 r_2)^{2n}}{r_1 - r_2}\left[r_1^{m-n} - r_2^{m-n}\right]$$

$$= F_{k,3n+3m} - (-1)^m F_{k,m-n} \quad (19)$$

IV. CONCLUSIONS

In this paper we developed the identities through the product of k-Fibonacci Numbers and k-Lucas Numbers. We tried to established connection formulas between k-Fibonacci and k-Lucas number by using Binet`s formula which can serve baseline for further research which involves product of Fibonacci and Lucas sequence.

ACKNOWLEDGMENT

We feel extremely indebted to our highly esteemed teacher Late Dr. Bijendra Singh, Prof. and Ex Head, School of Studies in Mathematics, Vikram University Ujjain (India) for the invaluable emotional strength and intellectual sharpness which have always anchored us.

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