On The Periodic Solution of the Helmholtz Equations Using the Modified Differential Transform Method

J. Peter Praveen, Koneru Lakshmaiah University, India, jppraveen17@kluniversity.in
B. Nageswara Rao, Koneru Lakshmaiah University, India, bnrao52@rediffmail.com

Abstract -**The modified differential transform method (MDTM) is formed the subject by a large number of publications. The MDTM is claimed to be an efficient method for obtaining periodic solution for the non-linear oscillatory systems. This paper examines the periodic solution of Helmholtz equation of motion having a non-odd restoring force function. The behavior of oscillations is expected to be different for the same magnitude of positive and negative amplitudes. The solution of the Helmholtz equation obtained utilizing the MDTM is unable to capture the unequal magnitudes of the positive and negative amplitudes for the specified frequency or the period. In addition to the presentation on the drawbacks in the MDTM, this paper recommends the harmonic balance method for obtaining accurate periodic solution of nonlinear Duffing oscillators.**

***Keywords*: Helmholtz equation, Harmonic balance method, Modified differential transform method, Phase diagram, Periodic solution**

1. INTRODUCTION

The differential transform method (DTM) is a semi-analytical-numerical technique for obtaining solution of linear and nonlinear differential equations. The method of approach resembles the Taylor series approximation. The governing differential equation along with the boundary conditions is transformed into a recurrence equation, which results into a system of algebraic equations. By solving these equations, one can obtain the coefficients in the power series solution. This method has been applied to solve structural dynamics problems, fluid flow problems, and so on [1-6]. However, the series solution obtained using the DTM is found to diverge with finite number of terms. In order to overcome such type of diverging solutions, the modified differential transform method (MDTM) is utilized by combining the differential transform method, Laplace transform and the Pade' approximant [7-25]. Many researchers have utilized the MDTM to obtain an approximate solution for non-linear Duffing oscillators. They claimed that their results are matching well with the numerical solutions using the fourth-order Runge-kutta integration scheme [6-12].

Motivated by the work of the above researchers, the periodic solution of the Helmholtz equation having a non-odd restoring force function is examined in this paper. The behavior of oscillations is expected to be different for the same magnitude of positive and negative amplitudes whereas the solution of the Helmholtz equation obtained using the MDTM could not capture the unequal magnitudes of positive and negative amplitudes for the specified period or the frequency. This paper presents the drawbacks of the MDTM while obtaining the periodic solution of the Helmholtz equation and recommends the harmonic balance method [26-30] for obtaining accurate periodic solution of nonlinear Duffing oscillators.

The organization of presentation is as follows. As usual the problem is solved using DTM with nine terms in the so called divergent series solution. Laplace transformation is applied to the series solution and expressed the result with [4/4] Pade approximant. The inverse transform of the [4/4] Pade approximant results the solution in terms of trigonometric functions, which avoid divergence representation of the series solution. In order to examine the adequacy of the solution, the phase diagram for the Helmholtz equation of motion is generated and found the range of the amplitudes for which the periodic solution exist. This study indicates the failure of the MDTM to provide the unequal magnitudes of the positive and negative amplitudes for the same period. The problem is solved using the higher order harmonics and demonstrated the expected behavior of the oscillations. This simple case study cautions the usage of MDTM for nonlinear oscillations.

2. SOLUTION OF THE HELMHOLTZ EQUATION

The objective of the present study is to examine the adequacy of the periodic solutions of Helmholtz equation utilizing the modified differential transform method (MDTM):

$$\frac{d^2y}{dt^2}+y+0.1\,y^2=0 \tag{1}$$

$$y=1,\ \frac{dy}{dt}=0 \ \text{ at } \ t=0 \tag{2}$$

The function $y(t)$ in (1) is represented by a power series (or Taylor series) around the initial point ($t=0$) as

$$y(t)=y(0+t)=y(0)+\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\frac{d^k y}{dt^k}\right]_{t=0}=\sum_{k=0}^{\infty}Y(k)\,t^k \tag{3}$$

Here $y(t)$ is the original function and $Y(k)$ is the transformed function. Using the initial conditions (2) in (3), one can find

$$Y(0)=[y]_{t=0}=1,\ Y(1)=\left[\frac{dy}{dt}\right]_{t=0}=0 \tag{4}$$

Taking the differential transform of (1), one can find the recurrence relation as

$$Y(k+2)=-\frac{1}{(k+1)(k+2)}\left\{Y(k)+0.1\sum_{l=0}^{k}Y(l)Y(k-l)\right\} \quad \text{for } k=0,\,1,\,2,........ \tag{5}$$

It should be noted that the recurrence relation (5) can be obtained by using (3) in (1), and comparing the coefficient of t^k.

From equations (4) and (5), one can find $Y(0)=1; Y(1)=0$; $Y(2)=-\frac{1.1}{2!}$; $Y(3)=0$; $Y(4)=\frac{1.32}{4!}$; $Y(5)=0$; $Y(6)=-\frac{2.31}{6!}$; $Y(7)=0$; $Y(8)=\frac{7.128}{8!}$; $Y(9)=0$; …… Using these values in (3), the following series solution is obtained:

$$y(t)=1-1.1\frac{t^2}{2!}+1.32\frac{t^4}{4!}-2.31\frac{t^6}{6!}+7.128\frac{t^8}{8!}-+.... \tag{6}$$

Applying the Laplace transformation to the series solution (6), one can find

$$\bar{y}(s)=L[y(t)]=\frac{1}{s}-\frac{1.1}{s^3}+\frac{1.32}{s^5}-\frac{2.31}{s^7}+\frac{7.128}{s^9}-+..... \tag{7}$$

The [4/4] Pade approximant to $\bar{y}(s)$ is

$$\bar{y}(s)=\frac{a_0+\sum_{k=1}^{4}a_k s^{-k}}{1+\sum_{k=1}^{4}b_k s^{-k}} \tag{8}$$

$$\Rightarrow\left\{1+\sum_{k=1}^{4}b_k s^{-k}\right\}\bar{y}(s)=a_0+\sum_{k=1}^{4}a_k s^{-k} \tag{9}$$

Use (7) in (9), and compare the coefficients of s^{-k} $(k=0,1,2,3,.....,9).$, one gets

$$a_0=0 \tag{10}$$

$$a_1=1 \tag{11}$$

$$b_1=a_2 \tag{12}$$

$$-1.1+b_2=a_3 \tag{13}$$

$$-1.1b_1+b_3=a_4 \tag{14}$$

$$1.32-1.1b_2+b_4=0 \tag{15}$$

$$1.32b_1-1.1b_3=0 \tag{16}$$

$$-2.31+1.32b_2-1.1b_4=0 \tag{17}$$

$$-2.31b_1-1.32b_3=0 \tag{18}$$

$$7.128-2.13b_2+1.32b_4=0 \tag{19}$$

Solving (15) and (17), one can find: $b_2 = 7.8$, $b_4 = 7.26$. From (13), $a_3 = 6.7$. Solving (16) and (18), one can find: $b_1 = b_3 = 0$. From (14), $a_4 = 0$. It should be noted that the above values do not satisfy (19). Hence, equation (8) is referred as approximant. Equation (8) can be written in the form

$\bar{y}(s) = L[Y(t)] = \dfrac{s^{-1} + 6.7s^{-3}}{1 + 7.8s^{-2} + 7.26s^{-4}} = \dfrac{s^3 + 6.7s}{s^4 + 7.8s^2 + 7.26} = \dfrac{s^3 + 6.7s}{\left(s^2 + \omega_1^2\right)\left(s^2 + \omega_2^2\right)}$, which can be in terms of partial fractions as

$$\vec{y}(s) = \frac{As}{s^2 + \omega_1^2} + \frac{Bs}{s^2 + \omega_2^2} \tag{20}$$

The constants in (20) are: $A = 0.996529$; $B = 0.003471$; $\omega_1^2 = 1.080425$; and $\omega_2^2 = 6.719575$. The inverse Laplace transformation to (20) provides the solution for (1) and (2) in the form

$$y(t) = A\cos(\omega_1 t) + B\cos(\omega_2 t) \tag{21}$$

Using the constants A, B, ω_1 and ω_2 in (21), one can find the solution obtained by Momani and Ertürk [7] using the MDTM as

$$y(t) = 0.996529\ \cos(1.03944\,t) + 0.003471\cos(2.59221\,t) \tag{22}$$

In order to examine the adequacy of the solution (22), the phase diagram for the Helmholtz equation of motion need to be generated, which will be presented below.

2.1 Generation of Phase Diagram

The restoring force function in the equation of motion (1) is

$$f(y) = y + 0.1y^2, \tag{23}$$

which is a quadratic polynomial. Hence, it is a non-odd function. The behavior of oscillations is different for positive and negative amplitudes. The singular points of the differential equation (1) in the phase diagram (i.e., $\dfrac{dy}{dt}$ versus y curve) obtained from the roots of $f(y)$ are $(0, 0)$ and $(-10, 0)$. The derivative of $f(y)$ with respect to y is: $f'(y) = 1 + 0.2y$. Since, $f'(0) = 1 > 0$ and $f'(-10) = -1 < 0$, the singular point $(0, 0)$ becomes a centre, whereas the other point $(-10, 0)$ becomes a saddle point. Defining the potential energy function, $I(y) = \int_0^y f(\xi)\,d\xi$, (which implies that $\dfrac{dI}{dy} = f(y)$), the equation of motion (1) can be written in the form

$$\frac{d^2y}{dt^2} + \frac{dI}{dy} = 0 \tag{24}$$

Multiplying (24) by $2\frac{dy}{dt}$ and applying the initial conditions (2), one gets after integration

$$\left(\frac{dy}{dt}\right)^2 + 2\{I(y) - I(1)\} = 0 \tag{25}$$

Equation (25) for the Helmholtz equation of motion (1) with the initial conditions (2) can be written in the form

$$15\left(\frac{dy}{dt}\right)^2 = (1-y)(y^2 + 16y + 16) \tag{26}$$

Equation (26) represents the phase diagram (i.e., $\frac{dy}{dt}$ versus y) for the differential equation (1) with initial conditions (2). The plot of $\frac{dy}{dt}$ versus y generated from (26) shows closed boundary, which implies the existence of the periodic solution. The magnitude of the positive and negative amplitudes of the non-linear oscillations from (26) are 1 and -1.0717 respectively, which implies non-symmetry of the phase diagram with respect to $\frac{dy}{dt}$ axis, whereas it is symmetric with respect to y -axis. Figure1 shows the phase diagram generated from (26) which indicates unequal magnitude of the positive and negative amplitudes. The results of (22) shown in Figure1 indicate equal magnitudes of positive and negative amplitudes.

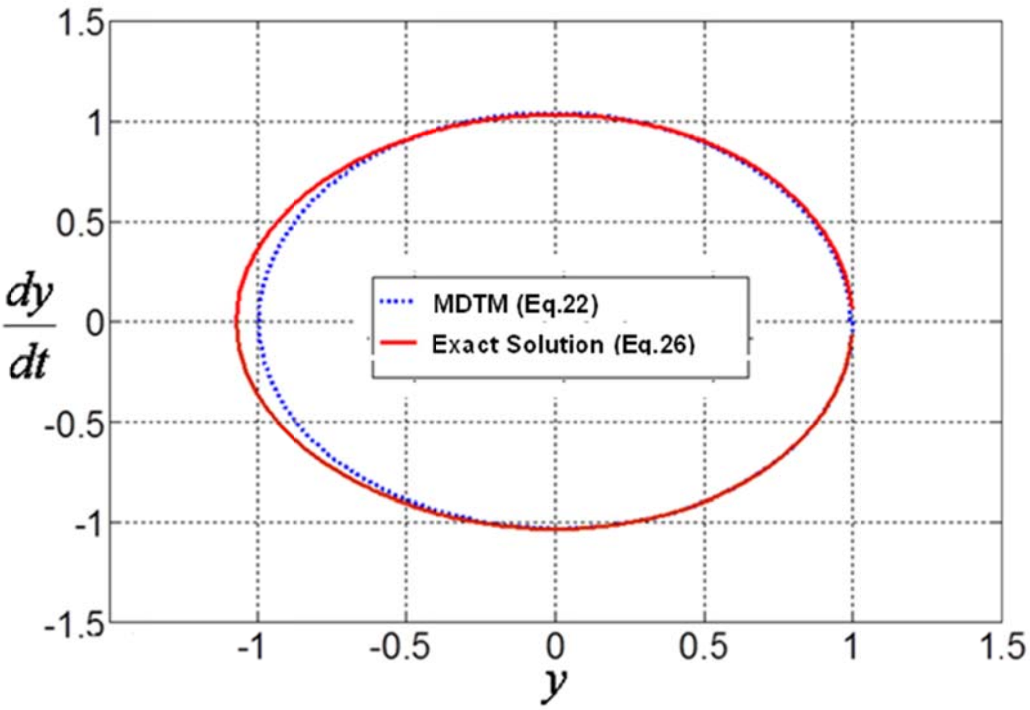

Figure1. Comparison of phase diagrams of Helmholtz equation generated from the solution of MDTM with the exact solution

The largest enclosed curve (viz., separatrix) in the phase-plane which passes just inside $(-10,\ 0)$ can be generated by replacing $I(1)$ in (25) with $I(-10)$ results

$$15\left(\frac{dy}{dt}\right)^2 = (5-y)(10+y)^2 \tag{27}$$

From (27), the range of amplitudes to obtain the periodic solution of the Helmholtz equation (1) is between -10 and 5. Outside this range, the periodic solution is not possible. If $y(0)=6$, then the phase diagram for the Helmholtz equation of motion (1) becomes

$$15\left(\frac{dy}{dt}\right)^2 = (6-y)(y^2+21y+126) \tag{28}$$

For the case $y(0)=-12$, the phase diagram for the Helmholtz equation of motion (1) can be generated from

$$15\left(\frac{dy}{dt}\right)^2 = (12+y)\ (4.68465-y)\ (7.68465+y) \tag{29}$$

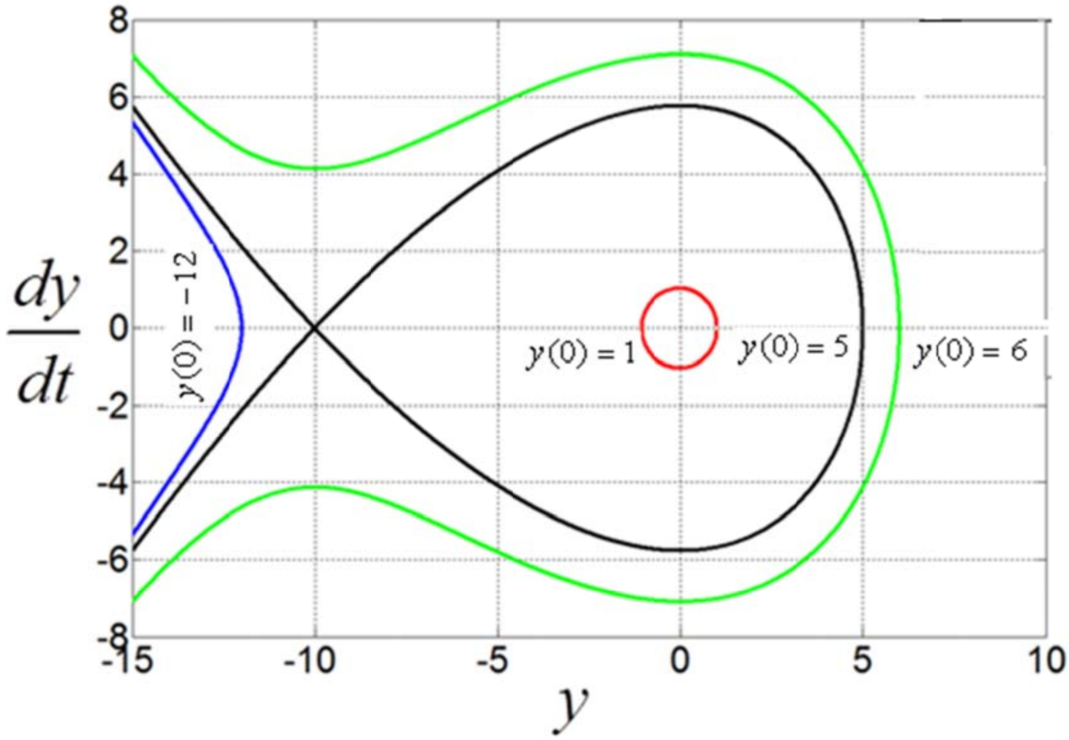

Figure2. Comparison of phase diagrams generated from (26) to (29) for the $y(0)$ values 1, 5, 6 and -12

Figure2 shows the comparison of phase diagrams generated from equations (26) to (29) for the $y(0)$ values 1, 5, 6 and -12. It should be noted that the phase diagram corresponding to $y(0) = 5$ represents the separatrix, whereas it is for $y(0) = 1$ represents the closed boundary having periodicity. The phase diagrams do not represent closed boundaries for $y(0) = 6$ and $y(0) = -12$ respectively. Figure2 confirms the validity of the amplitudes to obtain the periodic solution of the Helmholtz equation of motion (1). An attempt is made to obtain the solution of the Helmholtz equation of motion (1) with the initial conditions $y = 5, \frac{dy}{dt} = 0$ at $t = 0$. Using the MDTM, one can obtain the solution as

$$y(t) = 4.943389 \cos(1.186306\ t) + 0.056611 \cos(3.0972045\ t) \qquad (30)$$

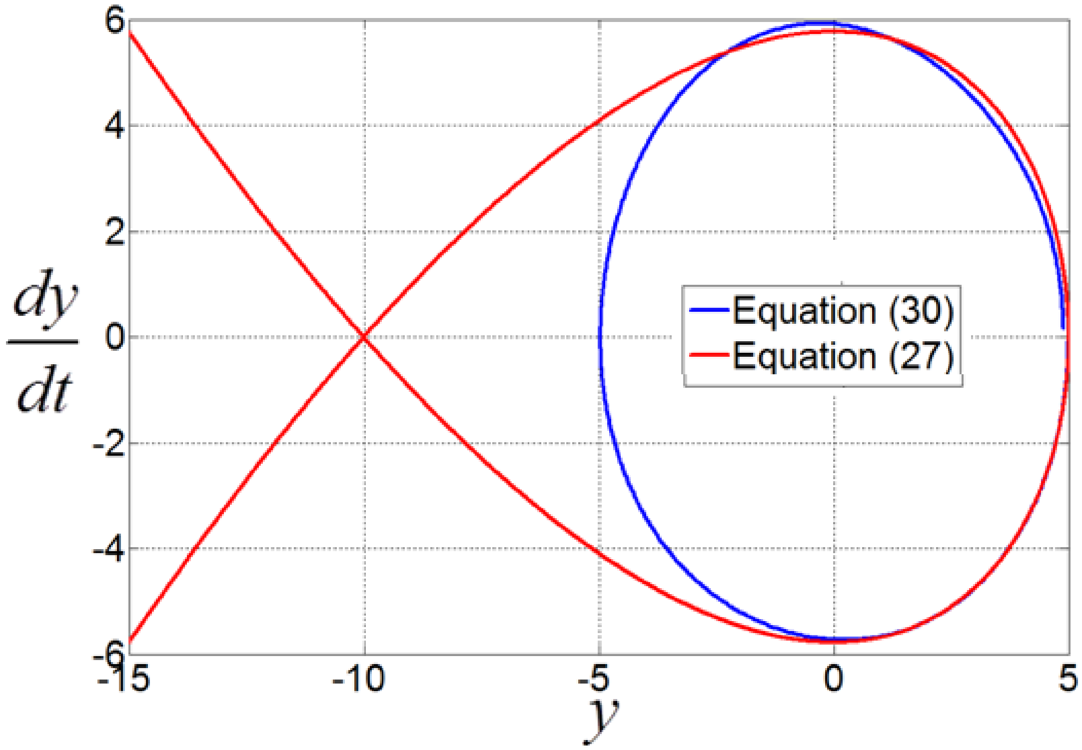

Figure3. Comparison of phase diagrams generated from (27) and (30) for $y(0)$ =5

It is very interesting to note from Figure3 that the phase diagram generated from (30) will have close magnitudes of positive and negative amplitudes, whereas the separatrix in Figure-2 indicates large difference in the magnitudes of the positive and negative amplitudes. The MDTM provides the solution accurately near to the region where the initial conditions are specified. An attempt is made to examine the behavior of nonlinear oscillations of the present problem utilizing the method of harmonic balance [26-30]. Analysis results based on the harmonic balance method are presented below.

2.2 Harmonic Balance Method

The function $y(t)$ in (1) which satisfies the initial conditions (2), is assumed in the form

$$y(t) = \alpha + (1-\alpha)\cos(\omega t) \tag{31}$$

After the use of trigonometric identities and application of the method of harmonic balance to retain only constant terms and terms of $\cos(\omega t)$, two equations are obtained. From these two equations, the constant, $\alpha = -0.05608$ and the frequency parameter, $\omega = 0.99438$. The solution of (1) and (2) obtained from the lowest order harmonics is

$$y(t) = -0.05608 + 1.0561\cos(0.99438\,t), \tag{32}$$

whereas, for the higher order harmonics, it is in the form

$$y(t) = -0.05396 + 1.0359\cos(0.9954\,t) + 0.018\cos(1.9908\,t) \tag{33}$$

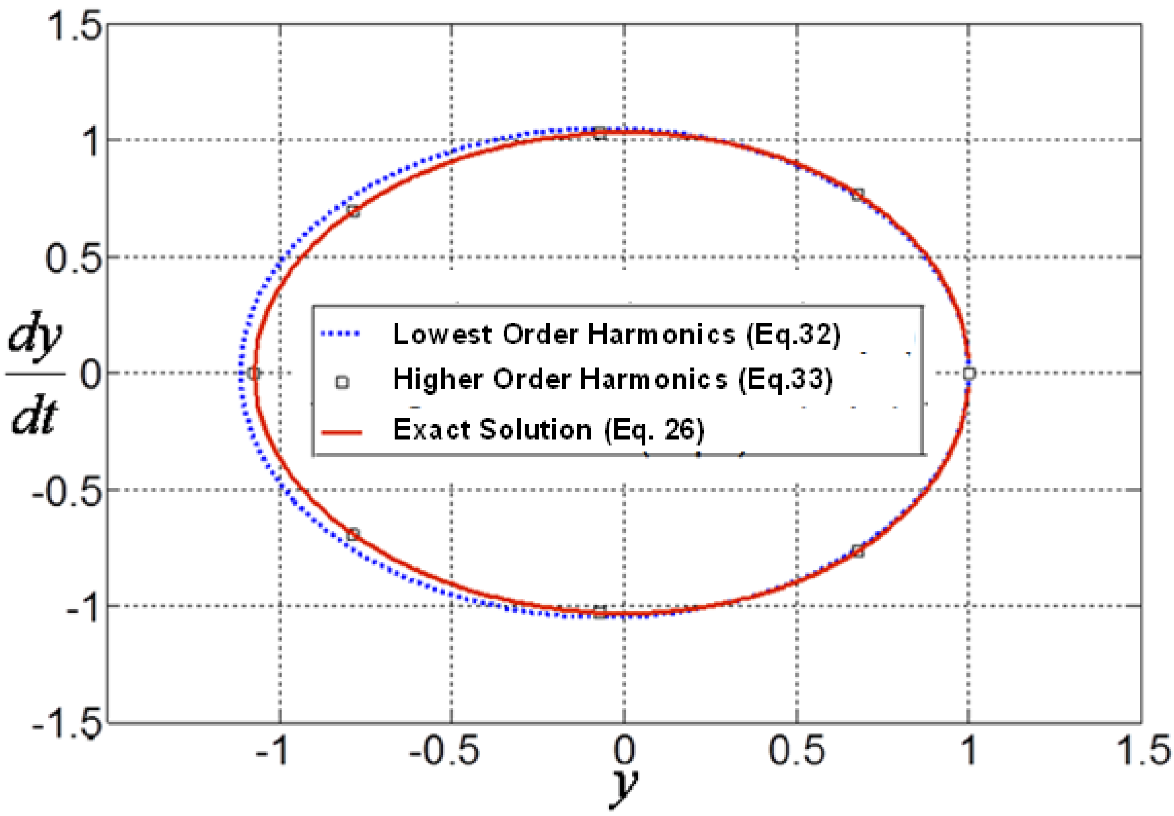

Figure4. Comparison of Phase diagrams of Helmholtz equation generated from the periodic solutions of Harmonic balance method with the exact solution.

Figure4 shows the comparison of the phase diagrams generated from (26), (32) and (33). Equations (32) and (33) show different magnitudes of the positive and negative amplitudes. Solution obtained from the higher order harmonic is very close to the actual phase diagram of the Helmholtz equation.

3. CONCLUDING REMARKS

The behavior of oscillations of the Helmholtz equation of motion (1) with initial conditions (2) can be easily understood through generation of phase diagrams. Hence, this problem is chosen in the present study to examine the adequacy of the modified differential transform method (MDTM). The solution procedure is briefly explained in the paper for clarity. Though the MDTM is claimed to be an efficient method for obtaining periodic solution for the non-linear oscillatory systems, the periodic solution of the Helmholtz equation of motion could not capture the unequal magnitudes of positive and negative amplitudes for the same period. Though the method seems to be simple, the calculations are tedious. Change of initial conditions demands repetition of the procedure as being followed in the numerical integration. Such complexity is not found while obtaining the periodic solution using the method of Harmonic balance. The range of amplitudes is identified from the restoring force function for which the periodic solution exists. The adequacy of that identified amplitude range is demonstrated through generation of phase diagrams. For the case of large amplitudes, large difference is noticed in the magnitudes of positive and negative amplitudes for the same period. MDTM is failed to provide such difference in the magnitudes of the positive and negative amplitudes for a simple Helmholtz equation of motion. Usage of higher order harmonics in the method of harmonic balance may provide realistic results. This case study clearly indicates that the modified differential transform method (MDTM) improves DTM truncated series solution, which may not yield the true periodic solution for the Helmholtz equation and cautions the usage of MDTM.

ACKNOWLEDGEMENT

The authors wish to thank the reviewers for their valuable comments/suggestions to improve the clarity of presenting the work.

REFERENCES

[1] V.S. Erturk and S. Momani, "Comparing numerical methods for solving fourth-order boundary value problems", Applied Mathematics and Computation, Vol.188, pp.1963-1968, 2007

[2] Q. Mao, "Design of shaped piezoelectric modal sensors for cantilever beams with intermediate support by using differential transformation method", Applied Acoustics, Vol.73, pp.144-149, 2012.

[3] M.M. Rashidi and E. Erfani, "Analytical method for solving steady MHD convective and slip flow due to a rotating disk with viscous dissipation and Ohmic heating", Engineering Computation, Vol.29, pp.562-579, 2012.

[4] X.H. Su and L.C. Zheng, "Approximate solutions to MHD Falkner-Skan flow over permeable wall", Applied Mathematics and Mechanics (English Edition), Vol.32, pp.401-408, 2011

[5] S. Hesam, A.R. Nazemi, and A. Haghbin, "Analytical solution for the Fokker-Plank equation by differential transform method", Scientia Iranika, Vol.19, pp.1140-1145, 2012

[6] F. Ayaz, "Solutions of the system of differential equations by differential transform method", Applied Mathematics and Computation, Vol.147, pp.547–567, 2004

[7] S. Momani and V.S. Ertürk, "Solutions of non-linear oscillators by the modified differential transform method", Computers & Mathematics with Applications, Vol.55, pp.833-842, 2008

[8] F. Mirzaee, "Differential transform method for solving linear and nonlinear systems of ordinary differential equations", Applied

Mathematical Sciences, Vol.5, No.70, pp.3465-3472, 2011
[9] H. Askari, Z. Saadatnia, D. Younesian, A. Yildirim, and M.K. Yazdi, “Approximate periodic solutions for the Helmhlotz-Duffing equation”, Computers & Mathematics with Applications, Vol. 62, pp.3894-3901, 2011.
[10] M. Merdan and A. Gökdogan, “Solution of non-linear oscillators with fractional nonlinearities by using the modified differential transformation method”, Mathematical and Computational Applications, Vol.16, No.3, pp.761-772, 2011.
[11] V.S. Ertürk, A. Yildirim, S. Momani, and Y. Khan, “The differential transform method and Pade’ approximants for a fractional population growth model”, International Journal of Numerical Methods for Heat & Fluid Flow, Vol. 22, pp.791-802, 2012.
[12] S. Nourazar and A. Mirzabeigy, “Approximate solution for nonlinear Duffing oscillator with damping effect using the modified differential transform method”, Scientia Iranica Transactions B: Mechanical Engineering, Vol.20, No.2, pp.364-368, 2013.
[13] S.F.M. Ibrahim and S.M. Ismail, “A new modification of the differential transform method for a SIRC influenza model”, International Journal of Computer Applications, Vol.69, No.19, pp.8-15, 2013
[14] K. Aruna and A.S.V. Ravi Kanth, “Two-dimensional differential transform method and modified differential transform method for solving nonlinear fractional Klein-Gordon equation”, National Academy Science Letters, Vol.37, No.2, pp.163-171, 2014
Doi: 10.1007/s40009-013-0209-0
[15] A. Jaghavi, A. Babaei and A. Mohammed Pour, “Solution of fractional Zakharov-Kuznetsov equations by reduced differential transform method”, Carpian Journal of Mathematical Sciences, Vol.4, No.1, pp.77-85, 2014.
[16] M. Gubbs, Y. Keskin and G. Oturanc, “Numerical solution of time dependent foam drainage equation”, Computational Methods for Differential Equations, Vol.3, No.2, pp.111-122, 2015
[17] A. Khambayat and N. Patil, “The numerical solution of differential transform method and the Laplace transform method for second-order differential equation”, International Journal of Mathematics and Computer Research, Vol.3, No.2, pp.871-875, 2015.
[18] O.O. Agboola, A.A. Opanuga, and J.A. Gbadeyam, “Solution of third-order ordinary differential equation using differential transform method”, Global Journal of Pure and Applied Mathematics, Vol.11, No.4, pp.2511-2516, 2015.
[19] M.M. Rashidi, L. Shameklu, E. Momoniat, and J. Qing, “Irreversibility analysis of magneto-hydrodynamic nano-fluid flow injected through a rotary disk”, Thermal Science, Vol.19, pp.s197-204, 2015.
[20] K. Kumari, P.K. Gupta and G. Shanker, “An exact solution of diffusion equation with boundary conditions by Pade-Laplace differential transform method”, International Journal of Mathematics and its Applications, Vol.3, No.4, pp.1-8, 2015.
[21] H.M. Abdelhafez, “Solution of excited non-linear oscillators under damping effects using the modified differential transform method”, Mathematics, Vol.4, No.11, 2016. doi:10.3390/math4010011
[22] S. Kan Zari, S. Ben Mariem and H. Sahraoui, “A reduced differential transform method for solving the advection and the heat-like equations”, Physics Journal, Vol.2, No.2, pp.84-87, 2016.
[23] S.O. Edeki, G.O. Akinlabi, and S.A. Adeosun, “Analytic and numeric solution of time-fractional linear Schrodinger equation”, Communications in Mathematics and Applications, Vol.7, No.1, pp.1-10, 2016
[24] B. Ghazanfari and P. Ebrahimi, “Differential transformation method for solving hybrid fuzzy differential equations”, journal of Hyperstructure, Vol.5, No.1, pp.69-83, 2016
[25] M. Najafgholipour and N. Soodbakhsh, “Modified differential transform method for solving vibration equations of MDOF systems”, Civil Engineering Journal, Vol.2, No.4, pp.123-139, 2016
[26] A. Venkateswara Rao and B. Nageswara Rao, “Some remarks on the harmonic balance method for mixed-parity non-linear oscillations”, Journal of Sound and Vibration, Vol.170, No.4, pp.571-576, 1994
[27] S.V.S. Narayana Murthy and B. Nageswara Rao, “Further comments on Harmonic balance: Comparison of equation of motion and energy methods”, Journal of Sound and Vibration, Vol.183, No.3, pp.563-565, 1995.
[28] R.E. Mickens, “Mathematical and numerical study of the Duffing-harmonic oscillator”, Journal of Sound and Vibration, Vol.244, pp.563-567, 2001
[29] B.S. Wu, W.P. Sun and C.W. Lim, “An Analytical approximate technique for a class of strongly non-linear oscillators”, International Journal of Nonlinear Mechanics, Vol.41, pp. 766-774, 2006
[30] M. Ghadaimi and H.D. Kaliji, “Application of the Harmonic balance method on nonlinear Equations”, World Applied Sciences Journal, Vol.22, No.4, pp. 532-537, 2013.