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Generalized Bivariate Fibonacci-Like Polynomials and Some Identities

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***Abstract*- In this paper, we introduce a generalized bivariate Fibonacci-Like polynomials sequence, from which specifying initial conditions the bivariate Fibonacci and Lucas polynomials are obtained. Also we define some properties of generalized bivariate Fibonacci-Like polynomials.**

***Keywords*- Generalized Bivariate Fibonacci-Like polynomials, Bivariate Fibonacci polynomials, and Binet's formula.**

I. INTRODUCTION

In [3], H. Belbachir and F. Bencherif generalize to bivariate polynomials of Fibonacci and Lucas, properties obtained for Chebyshev polynomials. They prove that the coordinates of the bivariate polynomials over appropriate basis are families of integers satisfying remarkable recurrence relations. [7], Mario Catalani define generalized bivariate polynomials, from which specifying initial conditions the bivariate Fibonacci and Lucas polynomials are obtained. In [5], K. Inoue and S. Aki investigate the properties of bivariate Fibonacci polynomials of order *k* in terms of the generating functions. In [8], Mario Catalani derive a collection of identities for bivariate Fibonacci and Lucas polynomials using properties of such polynomials when the variables x and y are replaced by polynomials. A wealth of combinatorial identities can be obtained for selected values of the variables. In [8], Mario Catalani derived many interesting identities for Fibonacci and Lucas Polynomials, these identities derived from a book of Professor Gould. In [1], D. Tasci, M. C. Firengiz and N. Tuglu define the incomplete bivariate Fibonacci and Lucas *p*-polynomials also generating function and properties of the incomplete bivariate Fibonacci and Lucas *p*-polynomials are given. In this paper, we present generalized bivariate Fibonacci-Like polynomials sequence and its properties like Catalan's identity, Cassini's identity or Simpson's identity and d'ocagnes's identity for generalized bivariate Fibonacci-Like polynomials.

II. GENERALIZED BIVARIATE FIBONACCI-LIKE POLYNOMIALS

We defined generalized bivararite Fibonacci-Like Polynomials $\{\mathrm{m}_{\mathrm{n}}(x)\}_{n=0}^{\infty}$ is defined by recurrence relation:

$$\mathrm{m}_{\mathrm{n}}(x,y)=\mathrm{axm}_{\mathrm{n-1}}(x)+y\mathrm{m}_{\mathrm{n-2}\,,}(\mathrm{x}),\mathrm{n}\geq 2$$

With initial conditions $\mathrm{m}_0(x,y)=2$ and $\mathrm{m}_1(x)=(2+\mathrm{b})x$ (2.1)

Where a and b are any integer.

For x=1 and y=b, we obtain generalized Fibonacci-Like Sequence for y=1, we obtain generalized Fibonacci-Like Polynomials.

The first few generalized bivararite Fibonacci-Like Polynomials are as follows.

$$m_1(x,y)=(2+b)x$$
$$m_2(x,y)=(2+b)ax^2+2y$$
$$m_3(x,y)=(2+b)a^3x^3+2axy+(2+b)xy$$
$$m_4(x,y)=(2+b)a^3x^4+2a^2x^2y+2(2+b)ax^2y+2y^2$$
$$m_5(x,y)=(2+b)a^4x^5+2a^3x^3y+3(2+b)a^2x^3y+4axy^2+(2+b)xy^2$$ and so on.

The characteristic equation of recurrence relation (2.1) is $t^2-axt-y=0$ this equation has two real roots;

$$\alpha=\frac{ax+\sqrt{a^2x^2+4y}}{2} \quad and \quad \beta=\frac{ax-\sqrt{a^2x^2+4y}}{2} \tag{2.2}$$

Also, $\alpha\beta=-y$ Binet's formula of Generalized bivararite Fibonacci-Like sequence is defined by

$$m_n(x,y)=C_2\alpha^n+D_2\beta^n \tag{2.3}$$

$$m_n(x,y)=C_2\left(\frac{ax+\sqrt{ax+4y}}{2}\right)^n+D_2\left(\frac{ax-\sqrt{ax+4y}}{2}\right)^n \tag{2.4}$$

Hear, $C_2=\frac{(2+b)x-2\beta}{\alpha-\beta}$ and $D_2=\frac{2\alpha-(2+b)x}{\alpha-\beta}$

$$C_2D_2=\frac{2(2+b)ax^2+4y-(2+b)^2x^2}{(\alpha-\beta)^2}$$

$$C_2\beta+D_2\alpha=(2a-2-b)x \tag{2.5}$$

$$C_2\beta^n+D_2\alpha^n=2a^2x^2+2y-(2+b)ax^2$$

Also,

$$\alpha\beta=-y,\alpha+\beta=m_0(x)=2 \quad \alpha-\beta=\sqrt{a^2x^2+4y},$$

$$\alpha^2+\beta^2=a^2x^2+2y \tag{2.6}$$

Generating function of generalized bivararite Fibonacci-Like Polynomials

$$\sum_{n=0}^{\infty}m_n(x,y)t^n=\frac{2+(2+b-2a)+xt}{(1-axt-yt^2)} \tag{2.7}$$

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Now Hyper geometric representation of generating function Generalized bivararite Fibonacci-Like Polynomials is given by $\sum_{n=0}^{\infty}\frac{m_n(x,y)}{n!}t^n=[2+(2+b-2a)xt]e^{axt}{}_2F_1[n+1:1,1,yt^2]$

Generating function (2.7), we have

$$\sum_{n=0}^{\infty}m_n(x,y)t^n=\frac{2+(2+b-2a)+xt}{(1-axt-yt^2)}$$

$$\sum_{n=0}^{\infty}m_n(x,y)t^n=[2+(2+b-2a)xt](1-axt-yt^2)^{-1}$$

$$=[2+(2+b-2a)xt][1-(ax+yt)t]^{-1}$$

$$=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}(ax+yt)^n t^n$$

$$=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}.t^n\sum_{k=0}^{n}\binom{n}{k}(ax)^{n-k}.(yt)^k$$

$$=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}.\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(ax)^{n-k}(y)^k.t^{k+n}$$

$$=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}.\sum_{k=0}^{\infty}\frac{(n+k)!}{k!n!}(ax)^n(y)^k.t^{2k+n}$$

$$=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}.\frac{(axt)^n}{n!}\sum_{k=0}^{\infty}\frac{(n+k)!}{n!k!}.(y)^k t^{2k}$$

$$\sum_{n=0}^{\infty}m_n(x,y)t^n=[2+(2+b-2a)xt]e^{axt}\sum_{k=0}^{\infty}\frac{(n+k)}{n!}.\frac{(yt^2)^k}{k!}$$

$$=[2+(2+b-2a)xt]e^{axt}\sum_{k=0}^{\infty}(n+1)_k\frac{(1)_k}{(1)_k}.\frac{(yt^2)^k}{k!}$$

Hence,

$$\sum_{n=0}^{\infty}\frac{m_n(x,y)}{n!}t^n=[2+(2+b-2a)xt]e^{axt}{}_2F_1[n+1:1,1,yt^2] \quad (2.8)$$

III. SOME IDENTITIES OF GENERALIZED BIVARIATE FIBONACCI-LIKE POLYNOMIALS

In this section, we present some identities of generalized bivararite Fibonacci-Like Polynomials. Catalan's; Cassini's; d'Ocagne's identities etc. by Binet's formula or explicit sum formula or generating function.

Theorem (3.1) (Explicit sum formula) Let $m_n(x, y)$ be the n^{th} Generalized bivararite Fibonacci-Like Polynomials. then

$$m_n(x,y)=2\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}(ax)^{n-2k}(y)^k+\left(\frac{2+b}{a}-2\right)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}(ax)^{n-2k}(y)^k \tag{3.1}$$

Proof: by generating function (2.7), we have

$$\sum_{n=0}^{\infty} m_n(x,y)t^n = \frac{2+(2+b-2a)+xt}{(1-axt-yt^2)}$$

$$=[2+(2+b-2a)xt][1-(ax+yt)t]^{-1}$$

$$=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}(ax+yt)^n t^n$$

$$=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}.t^n\sum_{k=0}^{n}\binom{n}{k}(ax)^{n-k}.(yt)^k=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}.\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(ax)^{n-k}(y)^k.t^{n+k}$$

$$=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}.\sum_{k=0}^{\infty}\frac{(n+k)!}{k!n!}(ax)^n(y)^k.t^{n+2k}$$

$$=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}.\sum_{k=0}^{\infty}\frac{(n+k)!}{n!k!}(ax)^n(y)^k t^{n+2k}$$

$$=[2+(2+b-2a)xt]\sum_{n=0}^{\infty}.\sum_{k=0}^{\frac{n}{2}}\frac{(n-k)!}{k!n-2k!}(ax)^{n-2k}(y)^k t^n$$

$$=\sum_{n=0}^{\infty}.\left[2\sum_{k=0}^{\frac{n}{2}}\frac{(n-k)!}{k!n-2k!}(ax)^{n-2k}(y)^k\right]t^n t$$

$$=\sum_{n=0}^{\infty}.\left[\left(\frac{2+b}{a}-2\right)\sum_{k=0}^{\frac{n}{2}}\frac{(n-k)!}{k!n-2k!}(ax)^{n-2k+1}(y)^k\right]t^{n+1}$$

Equating the coefficient of (t^n) on both sides, we obtained

$$m_n(x,y)=2\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}(ax)^{n-2k}(y)^k+\left(\frac{2+b}{a}-2\right)..\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}(ax)^{n-2k}(y)^k.$$

Theorem (3.2) **(sum of first n terms)** Sum of first n terms Generalized bivararite Fibonacci-Like Polynomials is given by

$$\sum_{k=0}^{n=0}m_n(x,y)=\frac{m_n(x,y)+ym_{n-1}(x,y)+(2a-2-b)x-2}{ax+y-1} \tag{3.2}$$

Proof: Using Binet's formula (2.4), we have

$$\sum_{k=0}^{n-1}m_k(x,y)=\sum_{k=0}^{n-1}\left[C_2\alpha^k+D_2\beta^k\right]$$

$$=C_2\sum_{k=0}^{n-1}\alpha^k+D_2\sum_{k=0}^{n-1}\beta^k$$

$$=C_2\left[\frac{1-\alpha^n}{1-\alpha}\right]+D_2\left[\frac{1-\beta^n}{1-\beta}\right]$$

$$=\frac{(C_2+D_2)-(C_2\beta+D_2\alpha)-(C_2\alpha^n+D_2\beta^n)+\alpha\beta(C_2\alpha^{n-1}+D_2\beta^{n-1})}{1-(\alpha+\beta)+\alpha\beta}$$

By Using (2.6), (2.4) and (2.5), we obtained required result.

Theorem (3.3) **(sum of first n terms with odd indices)** Sum of first n terms with odd indices of Generalized bivararite Fibonacci-Like Polynomials is given by

$$\sum_{k=0}^{n-1}m_{2k+1}(x,y)=\frac{m_{2n+1}(x,y)-y^2m_{2n-1}(x,y)+(2+b-2a)xy-(2+b)x}{a^2x^2-y^2+2y-1} \tag{3.3}$$

Proof: Using Binet's formula (2.4), we have

$$\sum_{k=0}^{n-1}m_{2k+1}(x,y)=\sum_{k=0}^{n-1}\left[C_2\alpha^{2k+1}+D_2\beta^{2k+1}\right]$$

$$=C_2\sum_{k=0}^{n-1}\alpha^{2k+1}+D_2\sum_{k=0}^{n-1}\beta^{2k+1}$$

$$=C_2\left[\frac{\alpha\left(1-\alpha^{2n}\right)}{1-\alpha^2}\right]+D_2\left[\frac{\beta\left(1-\beta^{2n}\right)}{1-\beta^2}\right]$$

$$=\frac{(C_2\alpha+D_2\beta)-\alpha\beta(C_2\beta+D_2\alpha)-(C_2\alpha^{2n+1}+D_2\beta^{2n+1})+(\alpha\beta)^2(C_2\alpha^{2n-1}+D_2\beta^{2n-1})}{1-(\alpha^2+\beta^2)+(\alpha\beta)^2}$$

By Using (2.6), (2.4) and (2.5), we obtained required result.

Theorem (3.4) (sum of first n terms with even indices) Sum of first n terms with even indices of Generalized bivararite Fibonacci-Like Polynomials is given by

$$\sum_{k=0}^{n-1} m_{2k}(x,y)=\frac{m_{2n}(x,y)-y^2 m_{2n-2}(x,y)-[(2+b)ax^2-2(a^2x^2y)]-2}{a^2x^2-y^2+2y-1} \tag{3.4}$$

Proof: Using Binet's formula (2.4), we have

$$\sum_{k=0}^{n-1} m_{2k}(x,y)=\sum_{k=0}^{n-1}\left[C_2\alpha^{2k}+D_2\beta^{2k}\right]$$

$$=C_2\sum_{k=0}^{n-1}\alpha^{2k}+D_2\sum_{k=0}^{n-1}\beta^{2k}$$

$$=C_2\left[\frac{\alpha\left(1-\alpha^{2n}\right)}{1-\alpha^2}\right]+D_2\left[\frac{\beta\left(1-\beta^{2n}\right)}{1-\beta^2}\right]$$

$$=\frac{(C_2+D_2)-\left(C_2\beta^2+D_2\alpha^2\right)-\left(C_2\alpha^{2n}+D_2\beta^{2n}\right)+(\alpha\beta)^2\left(C_2\alpha^{2n-2}+D_2\beta^{2n-2}\right)}{1-\left(\alpha^2+\beta^2\right)+(\alpha\beta)^2}$$

By Using (2.6), (2.4) and (2.5), we obtained required result.

Theorem (3.5) (Catalan's Identity) Let $m_n(x, y)$ be the n^{th} Generalized bivararite Fibonacci-Like Polynomials than

$$m_n^2(x,y)-m_{n+r}(x,y)m_{n-r}(x,y)=\frac{(-y)^{n-r}}{\left[(2+b)^2x^2-2(2+b)ax^2-4y\right]}\left[(2+b)xm_r(x,y)-2m_{r+1}(x,y)\right]^2, n>r\geq 1 \tag{3.5}$$

Proof: Using Binet's formula (2.4), to left hand side, we have

$$\left(C_2\alpha^n+D_2\beta^n\right)^2-\left(C_2\alpha^{n+r}+D_2\beta^{n+r}\right)\left(C_2\alpha^{n-r}+D_2\beta^{n-r}\right)$$

$$C_2D_2(\alpha\beta)^n-\left(2-\alpha^{-r}\beta^{-r}-\alpha^{-r}\beta^r\right)$$

$$\frac{[(2+b)^2x^2-2(2+b)ax^2-4y]}{(\alpha-\beta)^2}(-y)^{n-r}\left(\alpha^r-\beta^r\right)^2$$

$$[(2+b)^2x^2-2(2+b)ax^2-4y](-y)^{n-r}\left[\frac{\alpha^r-\beta^r}{\alpha-\beta}\right]^2$$

$$\frac{\alpha^r-\beta^r}{\alpha-\beta}=\frac{(2+b)xb_r(x,y)-2b_{r+1}(x,y)}{(2+b)^2x^2-2(2+b)ax^2-4y},$$

we obtained required identity.

As application of above identity we obtained following result.

Corollary (3.6) (Cassini's Identity): m_n (x,y) be the n^{th} term generalized bivariate Fibonacci-Like polynomial, then

$$m_m(x,y)m_{n+1}(x,y) - m_{m+1}(x,y)m_n(x,y) = (-y)^{n-1}\left[(2+b)^2x^2 - 2(2+b)ax^2 - 4y\right], n \geq 1 \quad (3.6)$$

Proof: Taking $r = 1$ in the Catalan's identity (3.5), the Proof is completed.

Theorem (3.7) (d'Ocagne's Identity): Let $m_n(x)$ be the n^{th} term generalized bivararite Fibonacci-Like polynomials, then

$$m_m(x,y)m_{n+1}(x,y) - m_{m+1}(x,y)m_n(x,y) = (-y)^n[(2+b)xm_{m-n}(x,y) - 2m_{m-n+1}(x,y)], m > n \geq 0 \quad (3.7)$$

Proof: Using Binet's formula (2.4), to left hand side, we have

$$= (C_2\alpha^m + D_2\beta^m)(C_2\alpha^{n+1} + D_2\beta^{n+1}) - (C_2\alpha^{m+1} + D_2\beta^{m+1})(C_2\gamma_2^n + D_2\delta_2^n)$$
$$= C_2D_2(\alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m - \alpha^n\beta^{m+1} - \alpha^{m+1}\beta^n)$$
$$= C_2D_2(\alpha\beta)^n\left[\beta(\alpha^{m-n} - \beta^{m-n}) - \alpha(\alpha^{m-n} - \beta^{m-n})\right] = -C_2D_2(-y)^n(\alpha - \beta)(\alpha^{m-n} - \beta^{m-n})$$
$$= \frac{[(2+b)x^2 - 2(2+b)ax^2 - 4y]}{(\gamma_2 - \delta_2)^2}(-y)^n(\alpha - \beta)(\alpha^{m-n} - \beta^{m-n})$$

$$by \frac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta} = \frac{[(2+b)xb_{m-n}(x,y) - 2b_{m-n+1}(x,y)]}{[(2+b)^2x^2 - 2(2+b)ax^2 - 4y]}$$

we obtained required identity.

Theorem (3.8) (Generalized Identity): Let m_n (x) be the n^{th} term generalized bivararite Fibonacci-Like polynomials, then

$$m_m(x,y)m_n(x,y) - m_{m-r}(x,y)m_{n+r}(x,y) = (-y)^{m-r}\frac{[(2+b)xm_r(x,y) - 2m_{r+1}(x,y)][(2+b)xm_{n-m+r}(x,y) - 2m_{n-m+r+1}(x,y)]}{[(2+b)x^2 - 2(2+b)ax^2 - 4y]}, n > m \geq r \geq 1 \quad (3.8)$$

Proof: Using Binet's formula (2.4), to left hand side, we have

$$= (C_2\alpha^m + D_2\beta^m)(C_2\alpha^n + D_2\beta^n) - (C_2\alpha^{m-r} + D_2\beta^{m-r})(C_2\alpha^{m+r} + D_2\beta^{m+r})$$
$$= C_2D_2(\alpha^r - \beta^r)\left[\frac{\alpha^m\beta^n}{\alpha^r} - \frac{\alpha^n\beta^m}{\beta^r}\right]$$
$$= C_2D_2\frac{(\alpha^r - \beta^r)}{(\alpha\beta)^r}\left[\alpha^m\beta^{n+r} - \alpha^{n+r}\beta^m\right]$$
$$= -C_2D_2(-y)^{m-r}(\alpha^r - \beta^r)(\alpha^{n-m+r} - \beta^{n-m+r})$$
$$\frac{[(2+b)^2x^2 - 2(2+b)ax^2 - 4y]}{(\alpha - \beta)^2}(-y)^{m-r}(\alpha^r - \beta^r)(\alpha^{n-m+r} - \beta^{n-m+r})$$

By $\dfrac{\alpha^r - \beta^r}{\alpha - \beta} = \dfrac{[(2+b)xb_r(x,y) - 2b_{r+1}(x,y)]}{[(2+b)^2x^2 - 2(2+b)ax^2 - 4y]}$,

$$and \frac{\alpha^{n-m+r} - \beta^{n-m+r}}{\alpha - \beta} = \frac{[(2+b)xb_{n-m+r}(x,y) - 2b_{n-m+r+1}(x,y)]}{[(2+b)^2x^2 - 2(2+b)ax^2 - 4y]}$$

we obtained required identity.

The identity (3.8) provides Catalan's identity, Cassini and d'Ocagne and other identities.

- If m=n, the Catalan's identity (3.5) is obtained.
- If m=n and $r = 1$, the Cassini's identity (3.6) is obtained.
- If n=m, m= $n+1$ and $r = 1$, d'Ocagne's identity (3.7) is obtained.

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